OPTIMAL SYSTEMS AND INVARIANT SOLUTIONS OF THE WAVE EQUATION ON SPHERICALLY SYMMETRIC SPACETIMES ADMITTING THE ISOMETRY GROUPS G7

BY

KHALEEL ANAYA

A Thesis Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

MAY 2017

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **KHALEEL ANAYA** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

Thesis Committee

Ahmad Y. Al-Dweik
Dr. Ahmad Al-Dweik (Adviser)

Hassan Azad
Dr. Hassan Azad (Member)

Dr. A. Laradji (Member)

Dr. Boubaker Smii (Member)

Dr. M. Tahir Mustafa (Member)

Dr. Hussain Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

13/6/17
Date

عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

To my father Ibrahim,

To my mother Lamees,

To my brothers and sisters Tamer, Samer, Mohammad, Yusra, Yasmeen, Masa and Maya.

ACKNOWLEDGMENTS

All praise is to Allah Who is the One, and His blessings and peace be upon him whom there is no messenger after him.

The foremost acknowledgments are due to my beloved grandfathers, grandmothers, my parents and brothers for their support, encouragement and prayers.

I would like to thank King Fahd University of Petroleum and Minerals (KFUPM) for giving me this opportunity to work on my Master program.

My deep appreciation goes to my thesis supervisor Dr. Ahmad Al-Dweik for his significant roles, encouragements and advices during this research work. He left a high impact on me.

Moreover, my deep thanks go to the thesis committee members for all of their efforts. I am also grateful to all faculty members and staff for providing a suitable research environment.

Finally, I sincerely thank all my relatives, teachers and friends.

TABLE OF CONTENTS

THESIS ABSTRACT

NAME:	Khaleel Anaya
TITLE OF STUDY:	Optimal systems and invariant solutions of the wave equation on spherically symmetric spacetimes admitting the isometry groups G_7
MAJOR FIELD:	Mathematics
DATE OF DEGREE:	May 2017

The focus of this work is to investigate the wave equations in spherically symmetric spacetimes admitting the isometry groups G_7. *Optimal systems up to three-dimensional subalgebras of the symmetry algebra will be constructed using an improved version of the systematic method introduced by Ovsiannikov [19]. The three-dimensional optimal system will be further used for classifying the group invariant solutions of the wave equations.*

ملخص الرسالة

الاسم: خليل عنايا

العنوان: المجموعة الكاملة المختلفة للحلول الثابتة لمعادلات الموجة على الشكل الكروي الذي له سبعة مجموعات تماثل

التخصص: الرياضيات

تاريخ التخرج: أيار 2017

ينصب تركيز هذا العمل على دراسة معادلات الموجة على الشكل الكروي بسبعة مجموعات تماثل. في هذه الرسالة، أوجدنا المجموعات الكاملة المختلفة للجبر التماثلي لغاية الدرجة الثالة بعد تعديل وتطوير طريقة تم عرضها مسبقا من قبل أوسيانيكوف [17]، ثم استخدمنا هذه المجموعة الكاملة المختلفة للجبر التماثلي من الدرجة الثالثة للحصول على مجموعة كاملة من الحلول المختلفة لمعادلات الموجة.

CHAPTER 1

PRELIMINARIES

In this chapter, we will introduce some terminology that is used in this thesis. Moreover, we refer the reader to [13] for concepts not defined in this thesis.

1.1 Introduction and Literature review

It was shown in [9, 6, 21] that spherically symmetric spacetimes belong to one of the following four classes according to their isometries and metrics.

- G_{10} corresponding to the static spacetimes Minkowski, de Sitter and anti de Sitter.
- G_7 corresponding to the static spacetimes Einstein and the anti Einstein universe, and one non-static spacetime.
- G_6 corresponding to the static spacetimes Bertotti-Robinson metric and two other metrics of Petrov type D, and six non-static spacetimes.

- G_4 is a class of metrics involving one or two arbitrary functions of one variable.

Azad et al [3] applied Lie group analysis to study the wave equation on the classes of static spherically symmetric spacetimes admitting the isometry groups G_{10} or G_7 or G_6. The Iwasawa decomposition for the symmetry algebras was obtained to partially classify non-conjugate solvable algebras. In this thesis, we complete the investigation started in [3] by finding the complete optimal system of subalgebras of dimension at most three and the corresponding invariant solutions. We expect these solutions to be of interest to mathematical physicists.

We can always construct a family of group invariant solutions obtained by using a subgroup of a symmetry group admitted by a given differential equation, as explained in [15]. Since there are infinitely many subgroups of symmetry group admitted by a given differential equation, listing of all the group invariant solutions is impossible. However, obtaining optimal systems-meaning conjugacy classes- of s-dimensional subgroups of the symmetry group and applying the optimal systems leads to an effective and systematic mechanism of classifying the group invariant solutions. This leads to non-similar invariant solutions under symmetry transformations.

Classifying the group invariant solutions by utilizing optimal systems is a significant application of Lie group and Lie symmetry methods to differential equations. The method was first introduced by Ovsiannikov [19]. He applied this method in classifying the invariant solutions of the one-dimensional gasdynamic equation [20]. Ibragimov extended this work to the two-dimensional adiabatic gas motions in his master thesis [14]. The main idea behind the method is discussed in detail in Ibragimov [15, 16], Olver [18] and Hydon [13].

As already stated above, we plan to extend the previous study [3] to various classes of spherically symmetric spacetimes. In more details we investigate wave equations admitting the isometry groups G_7 in the light of the following objectives:

1. Finding the Lie point symmetries.

2. Studying the structure of the Lie algebra of point symmetries generators.

3. Constructing an optimal systems of subalgebras of dimension one, two and three.

4. Classifying invariant solutions corresponding to the resulted three-dimensional optimal systems.

This thesis consists of the following chapters:

This chapter 1 provides the fundamental notations from the theory of Lie groups, Lie algebras, Lie symmetries of differential equations and invariant solutions.

Chapter 2,3 and 4 provide Lie point symmetries, Lie algebra structure, optimal system of orders at most three and invariant solutions of the wave equation on Einstein spacetime, anti-Einstein spacetime and non-static analog of Einstein spacetime, respectively.

1.2 Lie groups

Definition 1.1 *A **Lie group** is a group that is also a finite-dimensional smooth manifold, in which the group operations of multiplication and inversion are smooth maps.*

1.2.1 One-parameter group of transformations

Definition 1.2 *[8] Let $x = (x_1, x_2, ..., x_n)$ lie in a region $D \subseteq \mathbb{R}^n$. The set of transformations $\bar{x} = \varphi(x\ ;\varepsilon)$ defined for each x in D and parameter ε in set $S \subseteq \mathbb{R}$, with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a **one-parameter group of transformations** on D if the following hold:*

1. *For all $\varepsilon \in S$ the transformations are one-to-one onto D.*
2. *S with ϕ forms a group G.*
3. *For all $x \in D$, $\bar{x} = x$ when $\varepsilon = \varepsilon_0$ corresponding to the identity e, i.e., $\varphi(x\ ;\varepsilon_0) = x$.*
4. *If $\bar{x} = \varphi(x\ ;\varepsilon)$ and $\bar{\bar{x}} = \varphi(\bar{x}\ ;\delta)$, then $\bar{\bar{x}} = \varphi(x\ ;\phi(\varepsilon, \delta))$.*

1.2.2 Lie groups of transformations

Definition 1.3 *[8] A one-parameter group G of transformations $\bar{x} = \varphi(x\ ;\varepsilon)$ with the operation ϕ is a **one-parameter Lie group of transformations** if:*

1. *ε is a continuous parameter, i.e. S is an interval in $\mathbb{R}$.*

2. *φ is an infinitely differentiable function with respect to x in D and an analytic function of ε in S.*

3. *$\phi(\varepsilon, \delta)$ is an analytic function of ε and δ in S.*

Example 1.1 *Group of translations in the plane*

$\bar{x} = x + \varepsilon$

$\bar{y} = y$, *where* $\varepsilon \in \mathbb{R}$.

Here $\phi(\varepsilon, \delta) = \varepsilon + \delta$ *and the identity element corresponds to* $\varepsilon = 0$.

Example 1.2 *Group of scalings in the plane*

$\bar{x} = \varepsilon x$

$\bar{y} = \varepsilon^2 y$ *where* $0 < \varepsilon < \infty$.

Here $\phi(\varepsilon, \delta) = \varepsilon\delta$ *and the identity element corresponds to* $\varepsilon = 1$.

Through out this thesis, unless stated otherwise, we will assume that the identity element ε_0 is equal to 0.

1.3 Lie algebras

Definition 1.4 *A **Lie algebra** is a vector space V over some field F together with a binary operation $[\cdot, \cdot] : V \times V \to V$ called the **Lie bracket** or the **commutator** that satisfies the following axioms:*

1. *Bilinearity:*

   $[ax + by, z] = a[x, z] + b[y, z]$, $[z, ax + by] = a[z, x] + b[z, y]$, *where* $a, b \in F$.

2. *Alternativity:*

   $[x, x] = 0.$

3. *The Jacobi identity:*

   $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

   for all x, y, z *in* V*.*

Example 1.3 *Two main examples of Lie algebras are the algebra of* $n \times n$ *matrices over a field with operation* $[X, Y] = XY - YX$ *and the algebra of all vector fields on open sets of* $\mathbb{R}^n$ *with the commutator defined by* $[X, Y](f) = X(Y(f)) - Y(X(f))$*. Here, for a differentiable function* f*,* $X(f)$ *means the directional derivative of* f *in the direction of* X*.*

Definition 1.5 *Let* $\mathcal{L}$ *be a Lie algebra.*

- $\mathcal{L}$ *is called* ***abelian*** *if the Lie bracket is identically zero for all elements in* $\mathcal{L}$*.*
- *A vector subspace* $\mathcal{I} \subseteq \mathcal{L}$ *is an* ***ideal*** *if* $[x, y] \in \mathcal{I}$ *for all* $x \in \mathcal{I}$ *and* $y \in \mathcal{L}$*.*
- $\mathcal{L}$ *is* ***solvable*** *if the* ***derived algebra*** $D^n\mathcal{L} = 0$ *for a large enough* n*, where* $D^0\mathcal{L} = \mathcal{L}$ *and* $D^{n+1}\mathcal{L} = [D^n\mathcal{L}, D^n\mathcal{L}]$*.*
- $\mathcal{L}$ *is* ***semisimple*** *if there are no non-zero solvable ideals in* $\mathcal{L}$*.*
- $\mathcal{L}$ *is* ***simple*** *if it is non-Abelian, and contains 0 and* $\mathcal{L}$ *as the only ideals.*

One of the main uses of ideals is to construct homomorphic images of an algebra. Given an ideal $\mathcal{I}$ of a Lie algebra $\mathcal{L}$, we define a relation on $\mathcal{L}$ by $x \sim y$ if and only if $x - y \in \mathcal{I}$. This is an equivalence relation on $\mathcal{L}$. We denote the equivalence class containing x by $[x]$. This is the coset containing x. We define sum, product and scalar multiplication on these cosets by $[x] + [y] = [x + y]$, $[x][y] = [xy]$ and $\lambda[x] = [\lambda x]$. These are well defined operations. This makes the set of cosets into a homomorphic image of the algebra $\mathcal{L}$. We denote this image by $\mathcal{L}/\mathcal{I}$ and call the function $p : \mathcal{L} \to \mathcal{L}/\mathcal{I}$ defined by $p(x) = [x]$ the **projection map**.

Theorem 1.4 *Every finite dimensional Lie algebra $\mathcal{L}$ contains a unique largest solvable ideal denoted by $rad(\mathcal{L})$.*

Definition 1.6 *The ideal $rad(\mathcal{L})$ is called the* ***radical*** *of $\mathcal{L}$.*

Adjoint Representation A very important representation of an abstract Lie algebra $\mathcal{L}$ as an algebra of matrices is its **adjoint representation**. It is defined by $ad(x) : \mathcal{L} \to \mathcal{L}$ with $ad(x)(y) = [x, y]$, $x, y \in \mathcal{L}$.

Example 1.5 *Let $\mathcal{L}$ be a Lie algebra with basis $\{x_1, x_2\}$ and non-zero Lie bracket $[x_1, x_2] = x_1$. Then*

$$ad(x_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad ad(x_2) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Recall that a linear transformation $T : V \to V$, where V is a vector space, is called **semisimple** if its matrix representations are diagonalizable.

Definition 1.7 *If $\mathcal{L}$ is a Lie algebra, then a subspace S of $\mathcal{L}$ is said to be a* ***Lie subalgebra*** *if it is closed under the Lie bracket.*

Definition 1.8 *A Lie subalgebra H of a Lie algebra $\mathcal{L}$ is said to be a **Cartan subalgebra** if H is abelian and every element $h \in H$ is semisimple, we mean by a semisimple element that its adjoint representation is diagonalizable, and moreover H is maximal with these properties.*

Definition 1.9 *The **centralizer** of an element x in $\mathcal{L}$ denoted by $\mathcal{C}(x)$, is the set of all elements y in $\mathcal{L}$ that commute with x, i.e. $[x, y] = 0$ for all $y \in \mathcal{L}$.*

Definition 1.10 *Let S be a subalgebra. The **normalizer** of S in $\mathcal{L}$, denoted by $\mathcal{N}(S)$, is the set of all elements y in $\mathcal{L}$ such that $[y, x] \in S$ for all $x \in S$.*

In particular, if S is one-dimensional, say $\langle x \rangle$, then $\mathcal{N}(\langle x \rangle) = \{y : [y, x] = \lambda x \ for\ some\ constant\ \lambda\}$.

We need for later work an algorithmic procedure for calculating normalizers of subalgebras. Maple has a command to find normalizers of subalgebras, however this command gives wrong results if the basis of the subalgebra contains arbitrary parameters (see Example 1.7). To solve this problem, we will use the wedge product to determine the normalizer correctly [23]. The main properties of the wedge product are

- $x \wedge y = -y \wedge x$ where x, y are elements of a given vector space V. More generally, $x_1 \wedge \cdots \wedge x_r = sgn(\sigma)(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(r)})$ where σ is a permutation of $1, \ldots, r$ and $sgn(\sigma)$ is the sign of the permutation σ.

- If W is a subspace of V with basis $\{w_1, \ldots, w_r\}$, then a vector $v \in W$ if and only if $w_1 \wedge \cdots \wedge w_r \wedge v = 0$.

This gives us the following remark:

Remark 1.6 *Let* $\{X_1, \ldots, X_n\}$ *be a basis of a Lie algebra* $\mathcal{L}$, *let* V *be a space with basis* $\{X_{i_1} \wedge \cdots \wedge X_{i_d} : 1 \leq i_1 < \cdots < i_d \leq n\}$ *where* $1 \leq d \leq n$ *denoted by* $\bigwedge^d \mathcal{L}$ *and called the* d^{th} *exterior power of* $\mathcal{L}$. *Let* S *be a proper subalgebra with a basis* $\{Y_1, \ldots, Y_r\}$. *To find the normalizer of* S, *one needs to solve the system* $Y_1 \wedge \cdots \wedge Y_r \wedge [Y_j, y] = 0,\ j = 1, \ldots, r,$ *where* $y = \sum_{i=1}^{n} a_i X_i$.

Example 1.7 *Let* $\mathcal{L}$ *be a Lie algebra with basis* $\{X_1, \ldots, X_7\}$ *and the non-zero Lie brackets*

$$\begin{aligned}
&[X_1, X_4] = X_2, \quad [X_1, X_5] = X_3, \quad [X_1, X_6] = X_1, \quad [X_2, X_6] = X_2, \\
&[X_2, X_7] = -X_3, \quad [X_3, X_6] = X_3, \quad [X_3, X_7] = X_2, \quad [X_4, X_7] = -X_5, \\
&[X_5, X_7] = X_4.
\end{aligned} \tag{1.1}$$

Maple found that the normalizer of $\langle \alpha X_4 + X_6 \rangle, \alpha \in \mathbb{R}$ *to be* $\{X_4, X_5, X_6\}$ *while the correct answer is* $\{X_4, X_5, X_6\}$ *or* $\{X_4, X_5, X_6, X_7\}$ *depending on whether* α *is non-zero or zero. We obtained this result by solving the equation*

$$\alpha X_4 + X_6 \wedge [\alpha X_4 + X_6, y] = 0, \quad \text{where } y = \sum_{i=1}^{7} a_i X_i \tag{1.2}$$

for $a_i, i = 1, \ldots, 7$. *In general, Maple assumes that all the parameters appearing are non-zero. The use of wedge product streamlines the calculations when many parameters are involved.*

Definition 1.11 *The* ***center*** *of the Lie algebra* $\mathcal{L}$ *is the set of all elements* x *in*

$\mathcal{L}$ that commute with every element in $\mathcal{L}$. We denote it by $\mathcal{Z}(\mathcal{L})$.

Definition 1.12 *Let $\mathcal{L}$ be a Lie algebra. The **Killing form** on $\mathcal{L}$ is the symmetric bilinear form defined by*

$$\kappa(x, y) = tr(ad\ x \circ ad\ y)\ for\ x, y \in \mathcal{L}$$

where tr stands for the trace of a matrix.

From now on we will consider only algebras over the field of real numbers $\mathbb{R}$. All the Lie algebras considered will be subalgebras of matrices. We consider the space of $n \times n$ matrices as a Euclidean space with n^2 coordinates. In particular, the set of all non-singular matrices is open because its complement is defined by the determinant, which is a polynomial, to be zero and defines a closed set in $\mathbb{R}^{n^2}$.

Definition 1.13 *A set A in $\mathbb{R}^n$ is called **compact** if it is closed and bounded.*

Definition 1.14 *A **compact topological group** is a topological group whose topology is compact.*

Definition 1.15 *A subalgebra K of $n \times n$ matrices is called **compact** if the corresponding subgroup G generated by the exponential matrix $\exp(K)$ is compact, as a subgroup of all $n \times n$ non-singular matrices.*

As regards subgroups of matrices, we will consider only subgroups of $GL(n, \mathbb{R})$, the set of all non-singular $n \times n$ matrices. This is a topological group because the functions

$$\begin{aligned}
&m : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \to GL(n,\mathbb{R}) \\
&m(x,y) = xy \\
&i : GL(n,\mathbb{R}) \to GL(n,\mathbb{R}) \\
&i(x) = x^{-1}
\end{aligned} \tag{1.3}$$

are continuous.

Example 1.8 *The subalgebra*

$$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

is compact because the group generated by the exponentials of this one-dimensional algebra is

$$\left\{ \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} : \varepsilon \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}$$

This is clearly a compact subset of 2×2 *matrices: Here we are thinking of all* 2×2 *matrices as a copy of* $\mathbb{R}^4$*. On the other hand, the subalgebra*

$$\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

is not compact because the the group generated by the exponentials of this one-

dimensional algebra is

$$\left\{ \begin{pmatrix} \cosh\varepsilon & \sinh\varepsilon \\ \\ \sinh\varepsilon & \cosh\varepsilon \end{pmatrix} : \varepsilon \in \mathbb{R} \right\} \;=\; \left\{ \begin{pmatrix} x & y \\ \\ y & x \end{pmatrix} : x^2 - y^2 = 1, x \geq 1 \right\}$$

This is clearly a closed but not bounded subset of 2×2 *matrices.*

Remark 1.9 *Let A be any $n \times n$ matrix. The Lie algebra $\{X : XA + AX^t = 0\}$ is denoted by $so(p,q)$ if A is a $p+q$ diagonal matrix whose first p entries are 1 and the remaining entries are -1.*

1.3.1 Optimal system of subalgebras

1.3.1.1 One-dimensional optimal system of subalgebras

Let $\{X_1, \ldots, X_n\}$ be a basis of a Lie algebra $\mathcal{L}$, $V = \bigwedge^d \mathcal{L}$ be the d^{th} exterior power of $\mathcal{L}$ with the basis $\{X_{i_1} \wedge \cdots \wedge X_{i_d} : 1 \leq i_1 < \cdots < i_d \leq n\}$ where $1 < d < n$ and G be the one-parameter Lie group generated by $\{exp(\varepsilon X) : \varepsilon \in \mathbb{R}, X \in \mathcal{L}\}$.

Definition 1.16 *One-dimensional subspaces $< w_1 >$ and $< w_1' >$ in $\mathcal{L}$ are called **conjugate** or **equivalent** with respect to the group of transformations G if there is some transformation $g \in G$ and a non-zero constant k such that*

$$Ad_g(w_1) = k\ w_1', \tag{1.4}$$

where $Ad_g(w_1) = g w_1 g^{-1}$.

Let $\mathcal{L}$ be a solvable Lie algebra with basis $\{X_1, \ldots, X_k\}$. For $w_1 = \sum_{k=1}^{n} a_k \ X_k$, using the solvability of $\mathcal{L}$, its **general adjoint transformation matrix** $A(\varepsilon_1, ..., \varepsilon_n)$ is the product of the matrices of separate adjoint actions $\{A_i(\varepsilon_i)\}_{i=1}^{n}$

$$A := A(\varepsilon_1, ..., \varepsilon_n) = \prod_{i=1}^{n} A_i(\varepsilon_i), \quad (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n, \tag{1.5}$$

where each matrix $A_i(\varepsilon_i)$ corresponding to $Ad_{exp(\varepsilon_i X_i)}(w_1)$, $i = 1, ..., n$, through the mapping

$$w_1 := (a_1, a_2, ..., a_n) \xrightarrow{Ad_{exp(\varepsilon_i X_i)}} (a_1, a_2, ..., a_n)A_i(\varepsilon_i), \ i = 1, ..., n. \tag{1.6}$$

Hence, the equivalence between the subspaces $< w_1 >$ and $< w_1' >$ can be rewritten in matrix form as

$$KA = k \ \tilde{K}, \tag{1.7}$$

where $K = (a_1, ..., a_n)$, $\tilde{K} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ and k is a non-zero constant.

Remark 1.10 *[13] The matrix* $A_j(\varepsilon_j) = e^{\varepsilon_j C(j)}$, $j = 1, \ldots, n$, *where* $C(j)$ *is the matrix whose* $(i, k)^{th}$ *entries are given as* c_{ij}^k*. The constants* c_{ij}^k *are called the* ***structure constants*** *and defined as follows*

$$[X_i, X_j] = \sum_k c_{ij}^k X_k. \tag{1.8}$$

Definition 1.17 *One calls a list* $\{\tilde{X}_\alpha\}_{\alpha \in \mathcal{A}}$ *a* ***one-dimensional optimal system*** *for a Lie algebra* $\mathcal{L}$*, if it satisfies the conditions:*

i) completeness, i.e. any one-dimensional subalgebra of $\mathcal{L}$ is conjugate to some $\tilde{X}_\alpha$;

ii) inequivalence, i.e. $\tilde{X}_\alpha$ and $\tilde{X}_\beta$ are non-conjugate for distinct α and β.

Remark 1.11 *[13] A non-abelian Lie algebra has one or more invariant functions, where by invariant function, say $\phi(K)$, we mean the function ϕ that satisfies the condition*

$$\phi(Ke^{\varepsilon_j C(j)}) = \phi(K),\ j = 1, \ldots, n. \tag{1.9}$$

Differentiating both sides of (1.9) with respect to ε at $\varepsilon = 0$ leads to the condition

$$KC(j)\nabla\phi(K) = 0; \quad j = 1, \ldots, n \tag{1.10}$$

where

$$\nabla\phi(K) = \begin{pmatrix} \phi_1(K) \\ \vdots \\ \phi_n(K) \end{pmatrix}, \phi_i(K) = \frac{\partial\phi(K)}{\partial a_i},\ i = 1, \ldots, n.$$

By solving the system (1.10) we get the invariant function(s).

The method of finding an optimal system of a solvable subalgebra was discussed in detail in [19, 15]. The one-dimensional subalgebras Θ_1 can be streamlined by using invariant functions. This is done by using the formula given in [5] or [13] for the system of equations that give the invariant functions in the adjoint representation.

To illustrate this method, we give the following example:

Example 1.12 *In this example, we will explain the steps of finding a one-dimensional optimal system for the Lie algebra $\mathcal{L}$ with basis $\{X_1, \ldots, X_4\}$ and the following non-zero Lie brackets*

$$[X_1, X_4] = X_1, \quad [X_2, X_3] = X_1, \quad [X_2, X_4] = 3X_2, \quad [X_3, X_4] = -3X_2, \tag{1.11}$$

- *Step 1: Construct the matrices $C(j), j = 1, \ldots, 4$.*

$$C(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; C(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{pmatrix}$$

$$C(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}, \; C(4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- *Step 2: Find the adjoint matrices $A_j(\varepsilon_j), \; j = 1, \ldots, 4$;*

$$A_1(\varepsilon_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\varepsilon_1 & 0 & 0 & 1 \end{pmatrix}, \; A_2(\varepsilon_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\varepsilon_2 & 0 & 1 & 0 \\ 0 & -3\varepsilon_2 & 0 & 1 \end{pmatrix}$$

$$A_3(\varepsilon_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varepsilon_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3}{2}\varepsilon_3^2 & 0 & 3\varepsilon_3 & 1 \end{pmatrix}, \; A_4(\varepsilon_4) = \begin{pmatrix} e^{\varepsilon_4} & 0 & 0 & 0 \\ 0 & e^{3\varepsilon_4} & 0 & 0 \\ 0 & 1-e^{-3\varepsilon_4} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- *Step 3: Multiply the matrices $A_j(\varepsilon_j)$, $j = 1, \ldots, 4$ to get the matrix A;*

$$A = A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \begin{pmatrix} e^{\varepsilon_4} & 0 & 0 & 0 \\ \varepsilon_3 e^{\varepsilon_4} & e^{3\varepsilon_4} & 0 & 0 \\ -\varepsilon_2 e^{\varepsilon_4} & 1-e^{-3\varepsilon_4} & 0 & 0 \\ (\frac{3}{2}\varepsilon_3^2 - \varepsilon_1 - 3\varepsilon_2\varepsilon_3)e^{\varepsilon_4} & 3(\varepsilon_3 - \varepsilon_2)e^{3\varepsilon_4} & 0 & 1 \end{pmatrix}$$

Now, a general element of $\mathcal{L}$ is of the form

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$$

where $a_i \in \mathbb{R}$. We start by determining the set of elements $\tilde{X}$ that are conjugate to X under the general adjoint representation of X_1, X_2, X_3 and X_4 that is represented by the matrix A. Now, $\tilde{X}$ can be written as:

$$\tilde{X} = \tilde{a}_1X_1 + \tilde{a}_2X_2 + \tilde{a}_3X_3 + \tilde{a}_4X_4$$

where $\tilde{a}_i \in \mathbb{R}$. If there are such ε_j $(j = 1, \ldots, 4)$ that solve:

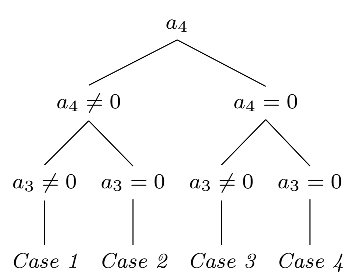
$$KA = \tilde{K} \tag{1.12}$$

where $K = (a_1, a_2, a_3, a_4)$ and $\tilde{K} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$, then $X, \tilde{X}$ are conjugate.

- *Step 4: Finding Invariant(s): Equation (1.10) becomes:*

$$\begin{pmatrix} a_4 & 0 & 0 & 0 \\ a_3 & 3a_4 & 0 & 0 \\ -a_2 & -3a_4 & 0 & 0 \\ -a_1 & 3(a_3 - a_2) & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(K) \\ \phi_2(K) \\ \phi_3(K) \\ \phi_4(K) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.13}$$

Solving the system (1.13), we get the solutions $\phi(a_1, a_2, a_3, a_4) = a_4$ and $\phi(a_1, a_2, a_3, a_4) = a_3$. Now we will consider two cases, either $a_4 = 0$ or $a_4 \neq 0$. Substituting these cases back into system (1.13), we will get another invariant. By repeating this step, we will get the following tree



The values of $\varepsilon_j, j = 1, .., 4$ and the representative element for each case can be given as

1. *Case 1:* $\left\{\varepsilon_1 = \frac{3a_1\, a_4 - a_2\, a_3}{3a_4^2}, \varepsilon_2 = \frac{a_2}{3a_4}, \varepsilon_3 = 0, \varepsilon_4 = 0\right\}$, *the representative element is $X_3 + \alpha X_4$, $\alpha \neq 0$.*

2. *Case 2:* $\left\{\varepsilon_1 = \frac{a_1}{a_4}, \varepsilon_2 = \frac{a_2}{3a_4}, \varepsilon_3 = 0, \varepsilon_4 = 0\right\}$, *the representative element is X_4.*

3. *Case 3:* $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a1}{a2}, \varepsilon_4 = 0\right\}$, *the representative element is* $X_2 + \alpha X_3,\ \alpha \neq 0.$

4. *Case 4:* $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a_1}{a_2}, \varepsilon_4 = 0\right\}$, *the representative element is* X_2.

Therefore, the optimal system of generators is

$$\langle X_2\rangle,\ \langle X_2 + \alpha X_3\rangle,\ \langle X_3 + \alpha X_4\rangle,\ \langle X_4\rangle,\ \alpha \neq 0 \tag{1.14}$$

1.3.1.2 Higher-dimensional optimal system of subalgebras

Definition 1.18 *Two-dimensional subspaces* $< w_1, w_2 >$ *and* $< w'_1, w'_2 >$ *in* $\mathcal{L}$ *are called* ***equivalent*** *with respect to the group* G *if there is some transformation* $g \in G$ *and some constants* $\{k_{11}, k_{12}, k_{21}, k_{22}\}$ *with* $det[k_{ij}] \neq 0$ *such that*

$$\begin{aligned} Ad_g(w_1) &= k_{11}w'_1 + k_{12}w'_2, \\ Ad_g(w_2) &= k_{21}w'_1 + k_{22}w'_2. \end{aligned} \tag{1.15}$$

Remark 1.13 *Similarly, if* $w_1 = \sum\limits_{k=1}^{n} a_k\ X_k$ *and* $w_2 = \sum\limits_{k=1}^{n} b_k\ X_k$, *then the system (1.15) can be rewritten in matrix form as*

$$\begin{pmatrix} a_1 & a_2 & .. & a_n \\ b_1 & b_2 & ... & b_n \end{pmatrix} A = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} a'_1 & a'_2 & ... & a'_n \\ b'_1 & b'_2 & ... & b'_n \end{pmatrix}, \tag{1.16}$$

where $\{k_{11}, k_{12}, k_{21}, k_{22}\}$ *are constants with* $det[k_{ij}] \neq 0$.

Remark 1.14 *Equivalently, we can say that Definition 1.18 implies that the one-dimensional subspaces* $< w_1 \wedge w_2 >$ *and* $< w'_1 \wedge w'_2 >$ *in* $\bigwedge^2 \mathcal{L}$ *are equivalent since*

$$Ad_g(w_1 \wedge w_2) = (k_{11}k_{22} - k_{21}k_{12})\ (w'_1 \wedge w'_2), \tag{1.17}$$

where $Ad_g(w_1 \wedge w_2) = Ad_g(w_1) \wedge Ad_g(w_2)$.

Remark 1.15 *As a summary, the one-dimensional subspaces* $< w >$ *and* $< w' >$ *in* $\bigwedge^2 \mathcal{L}$ *are equivalent with respect to the group* G *if there are* $\varepsilon_1, ..., \varepsilon_n$ *and a non-zero constant* k *such that*

$$(a_1, a_2, ..., a_m)A'(\varepsilon_1, ..., \varepsilon_n) = k\ (a'_1, a'_2, ..., a'_m), \tag{1.18}$$

where $m = \dim \bigwedge^2 \mathcal{L} = \binom{n}{2}$ *and the* $m \times m$ *matrix* $A'(\varepsilon_1, ..., \varepsilon_n)$ *can be constructed from the matrix* A *as follows:*

Let $\{X_1, \ldots, X_n\}$ *be a basis of a Lie algebra* $\mathcal{L}$, $V = \bigwedge^2 \mathcal{L}$ *be the second exterior power of* $\mathcal{L}$ *with the basis* $B = \{X_i \wedge X_j \mid 1 \leq i < j \leq n\}$ *with the dimension* $m = \binom{n}{2}$ *and* $A(i)$ *be the* i^{th} *row of the matrix* A *given in (1.5) written in terms of the basis of* $\mathcal{L}$.

Define $\psi : B \to M_{1\times m}$ *such that* $\psi(X_i \wedge X_j) = A(i) \wedge A(j)$ *written in terms of the basis of* $\bigwedge^2 \mathcal{L}$. *Then*

$$A' = \begin{pmatrix} \psi(B_1) \\ \vdots \\ \psi(B_m) \end{pmatrix}_{m\times m}$$

Similarly, equivalence of all higher-dimensional subspaces can be defined.

Definition 1.19 *A list of n-dimensional subalgebras $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is called an* ***n-dimensional optimal system*** *for a Lie algebra $\mathcal{L}$, if it satisfies the conditions: completeness and inequivalence.*

Higher-dimensional optimal system of solvable subalgebras:

To compute higher-dimensional optimal systems Θ_t of t-dimensional solvable subalgebras of the algebra $\mathcal{L}$, the **expansion method** can be employed [19, 15]. Even in the case of non-solvable Lie algebra, the expansion method can still be used to find the solvable subalgebras. The expansion method depends on the following theorem.

Theorem 1.16 *If X is a solvable Lie algebra then there is a chain of subalgebras X_i of dimension i with X_i an ideal in $X_{i+1}, 0 \leq i \leq d-1$, where d is the dimension of X.*

Proof. To see this, notice that if X is solvable, then X', the derived algebra of X, is a proper subalgebra of X and X/X' is abelian. Therefore, any subspace of X/X' is a subalgebra of X/X'. By using induction, we can find a chain of subalgebras of X' with the property that X_i is an ideal in $X_{i+1},\ 0 \leq i \leq d-1$ where $d = dim(X')$. Let $\bar{e_1}, \bar{e_2}, ..., \bar{e_k}$ be a basis of X/X'. Then a chain for X' can be completed to a chain for X by the inclusions $X' \subset \langle X', e_1 \rangle \subset \cdots \subset \langle X', e_1, ..., e_k \rangle = X$ ∎

This makes it possible to get an optimal system Θ_t of t-dimensional subalgebras using an optimal system Θ_{t-1} by expanding every $(t-1)$-dimensional subalgebra from Θ_{t-1} to a t-dimensional subalgebra and by elimination of conjugate subalgebras.

The expansion method is revised and improved to a systematic method by using the normalizers and their associated quotient algebras. Namely, for every $X \in \Theta_{t-1}$, we find the normalizer $\mathcal{N}(X)$. In case the quotient algebra $\mathcal{N}(X)/X$ is non-zero, we find a one-dimensional optimal system in $\mathcal{N}(X)/X$ for every $X \in \Theta_{t-1}$ by considering the invariants of the adjoint representation of $\mathcal{N}(X)/X$. Among the constructed optimal systems of $\mathcal{N}(X)/X$ for every $X \in \Theta_{t-1}$, we may still have repetitions in their preimages in $\mathcal{L}$. These repetitions can be removed using the wedge product as in (1.17). Removing the repetitions provides an optimal system Θ_t. Enumeration of all non-conjugate subalgebras of $\mathcal{L}$ can finally be done through consecutive choices of the values of t from 1 till $dim(\mathcal{L})$.

Three-dimensional optimal systems of non-solvable subalgebras:

Definition 1.20 *Let $\mathcal{L}$ be a Lie algebra and A, I be two subalgebras where A, I span $\mathcal{L}$ as a vector space with $A \cap I = 0$ and I is an ideal of $\mathcal{L}$. Then we say that $\mathcal{L}$ is a* ***semidirect sum*** *of A and I, and we write $\mathcal{L} = A \oplus_s I$. Moreover, in case A and I commute the semidirect sum is called a* ***direct sum****.*

In order to get the non-solvable subalgebras, one can successfully employ the following theorem:

Theorem 1.17 [15] (Levi Theorem) Let $\mathcal{L}$ be a finite-dimensional Lie algebra $\mathcal{L}$

and $\mathscr{R}(\mathcal{L})$ its unique radical. Then $\mathcal{L}$ is the semidirect sum

$$\mathcal{L} = \mathscr{R}(\mathcal{L}) \oplus_s \mathscr{L} \tag{1.19}$$

of $\mathscr{R}(\mathcal{L})$ and a semisimple subalgebra $\mathscr{L}$ of $\mathcal{L}$, which is not uniquely determined. (1.19) is called a **Levi decomposition** and $\mathscr{L}$ is said to be a **Levi subalgebra** or **Levi factor**.

In case the radical is itself the center of the Lie algebra, the optimal systems can be determined simply by finding optimal systems of solvable subalgebras of the semisimple part and the radical part independently. Finally, joining them using the procedure described in the following chapters provides the optimal systems of the full algebra.

1.3.1.3 Structure of semisimple Lie Algebras

The structure of a semisimple Lie algebra is determined by its roots. For more details the reader is referred to [1]; see also [12], [11] and [17].

Definition 1.21 *Let C be a Cartan subalgebra of a semisimple Lie algebra $\mathcal{L}$. A non-zero vector $v \in \mathcal{L}^{\mathbb{C}} := \mathcal{L} + i\mathcal{L}$ such that $[h, v] = \lambda(h)v$ for all $h \in C$ is called a* ***root vector*** *and the corresponding linear function λ is called a* ***root of the Cartan algebra*** *C.*

In general, the roots will be complex-valued. In the following argument, we will use the notion of positive roots, so one needs to define what it means for a complex valued root to be positive.

Definition 1.22 *A complex number $z = a + ib,\ a, b \in \mathbb{R}$ is **positive** if either its real part a is positive or $a = 0$ but its imaginary part b is positive.*

Fix a basis $h_1, \ldots, h_r$ of a Cartan algebra C. A non-zero root λ is **positive** if the first non-zero number $\lambda(h_i)$ is a complex positive number. Otherwise, it is called a **negative root**. Positive roots which are not a sum of two positive roots are called **simple roots**.

The well known software Maple is able to find the root space decompositions of Lie algebras of fairly high dimensions by using the command "RootSpaceDecomposition(C)" where C is a list of vectors in a Lie algebra, defining a Cartan subalgebra.

The Cartan algebra is picked up using an algorithm due to de Graaf [11]. However, one gets better coordinates for computation if one chooses a Cartan algebra by enlarging a given diagonalizable subalgebra to a Cartan subalgebra following the algorithms given in [1]. We need in this thesis only a special case of these algorithms to compute the Cartan subalgebras. We first compute the Killing form of the Lie algebra. If it is negative definite, pick any non-zero element X and compute its centralizer. By a negative definite matrix, we mean a matrix which is equal to its conjugate transpose and its eigenvalues are strictly negative. If the centralizer of X, $\mathcal{C}(X)$, is self centralizing, i.e. $\mathcal{C}(\mathcal{C}(X)) = \mathcal{C}(X)$, then $\mathcal{C}(X)$ is the Cartan subalgebra. Otherwise, we can find an element Y in the centralizer of X such that X and Y are linearly independent. Continue this procedure with the abelian algebra $\langle X, Y \rangle$ till a self centralizing subalgebra is reached. The obtained algebra is the Cartan algebra because it is abelian and every element is diagonalizable.

On the other hand, if the Killing form is not negative definite and a maximal compact subalgebra is known, say K, then computing a Cartan subalgebra C of K using the procedure explained in the previous paragraph for compact algebras and the centralizer of C in the full Lie algebra gives us the required Cartan algebra.

The main use of Cartan algebras is to find all the maximal solvable subalgebras [4]. In case the Lie algebra $\mathcal{L}$ is compact, a Cartan algebra is, up to conjugacy, the only maximal solvable subalgebra. This follows from Lie's theorem on solvable algebras [12].

There is a solvable subalgebra B with real eigenvalues in the adjoint representation of $\mathcal{L}$ with the property that any other solvable algebra with real eignvalues in the adjoint representation is conjugate to a subalgebra of B.

In [7], it is found that the algebra B can be constructed algorithmically by using positive roots of a given maximally real Cartan subalgebra, where by maximally real Cartan subalgebra we mean a Cartan algebra whose real part has maximal possible dimension. In case the Killing form is not negative definite, any Cartan algebra is a sum of two subalgebras such that one of them has all real eigenvalues in the adjoint representation in $\mathcal{L}$ and the other has all purely imaginary eigenvalues in the adjoint representation in $\mathcal{L}$. We call the first subalgebra the real part of the Cartan subalgebra and the second subalgebra the compact part of the Cartan subalgebra. Let N be the algebra consisting of the real and imaginary parts of the positive root vectors for the given maximally real Cartan subalgebra. Then the algebra $B = A + N$ where A is the real part of the maximally real Cartan subalgebra has the property that every solvable algebra with real eigenvalues in the adjoint representation is conjugate to a subalgebra of B. Moreover, all maximal solvable algebras which are non-abelian can be obtained by computing normalizers of subalgebras of N. In more detail, we consider conjugacy classes of subalgebras of N. If X is a representative of such a class, we compute the normalizer of X and its Levi decomposition. We keep only those X in which the normalizer of X has Levi decomposition $\mathcal{N}(X) = S + \mathcal{R}(\mathcal{N}(X))$, where the semisimple part has a compact Cartan subalgebra. If T is this compact Cartan subalgebra, then $T + \mathcal{R}(\mathcal{N}(X))$ is a maximal solvable subalgebra and all such are obtained in this way.

Example 1.18 *The subalgebra*

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

is a Cartan subalgebra of $sl(2, \mathbb{R})$ *with roots* $-2, 0, 2$ *and*

$$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

is a Cartan subalgebra of $sl(2, \mathbb{R})$ *with roots* $0, -2i, 2i$, *where* $sl(2, \mathbb{R})$ *is the set of all* 2×2 *matrices with trace zero.*

Example 1.19 *The subalgebra*

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

is a Cartan subalgebra of $so(3)$ *with roots* $0, i, -i$*, where* $so(3)$ *is the set of all skew symmetric* 3×3 *matrices.*

It is a classical fact that any non-solvable three-dimensional subalgebra is isomorphic to either $sl(2,\mathbb{R})$ or $so(3)$; see e.g. [2]. Therefore, one can construct a three-dimensional optimal system of non-solvable subalgebras by finding copies of $so(3)$ and $sl(2,\mathbb{R})$. In order to find such copies in a semisimple Lie algebra S, we will introduce the following algorithms which are based on the canonical relations for $so(3)$

$$[X, A] = Y, \quad [A, Y] = X, \quad [Y, X] = A, \tag{1.20}$$

and $sl(2,\mathbb{R})$

$$[A, B] = 2B, \quad [A, Y] = -2Y, \quad [B, Y] = A. \tag{1.21}$$

i) To find the copies of $so(3)$, we start with an element A of the one-dimensional optimal system of S whose non-zero eigenvalues in the adjoint representation are purely imaginary. By scaling we may assume that this eigenvalue is i. Let $X + iY$ be the eigenvector of A corresponding to the eigenvalue i. If $[X, Y] = \lambda A$ for some negative constant λ, then the algebra $\langle A, X, Y\rangle$ forms a copy of $so(3)$. Applying this algorithm for all elements in the one-dimensional optimal system gives us the copies of $so(3)$.

ii) To find the copies of $sl(2,\mathbb{R})$, we need to start with an element of the two-dimensional optimal system of non-abelian subalgebras. If $\langle A, B\rangle$ is such an algebra with $[A,B] = cB$ for some non-zero constant c, find the eigenvectors of ad$A$ corresponding to the eigenvalue $-c$ if it exists. We reject $\langle A, B\rangle$ if there is no such eigenvalue. Otherwise, let Y be an eigenvector of $\mathrm{ad}(A)$ with eigenvalue $-c$. If the commutator $[B,Y]$ is a non-zero multiple of A, then $\langle A, B, Y\rangle$ is a copy of $sl(2,\mathbb{R})$.

1.4 Lie symmetries

1.4.1 Infinitesimal transfomations and generators

In this section, we present some standard results, more details can be found in [8].

Definition 1.23 ***(Infinitesimal transformations)*** *Consider a one-parameter (ε) Lie group of transformation $\bar{x} = \varphi(x\ ;\varepsilon)$ with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding $\bar{x}$ about $\varepsilon = 0$, one gets,*

$$\bar{x} = x + \varepsilon \frac{\partial \bar{x}}{\partial \varepsilon}\bigg|_{\varepsilon=0} + O(\varepsilon^2) \tag{1.22}$$

where $\frac{\partial \bar{x}}{\partial \varepsilon}\big|_{\varepsilon=0} = \xi(x)$. The transformation $\bar{x} = x + \varepsilon\ \xi(x)$ is called the ***infinitesimal transformation*** *of the Lie group of transformations and the components*

$\xi(x)$ *are called the* ***infinitesimals*** *of the transformation.*

Theorem 1.20 ***(First Fundamental Theorem of Lie)*** *There exists a parametrization* $\tau(\varepsilon)$ *such that the Lie group of transformations* $\bar{x} = \varphi(x\ ;\varepsilon)$ *can be represented as the solution of an initial value problem for the system of first order ordinary differential equations given by*

$$\frac{d\ \bar{x}}{d\ \tau} = \xi(\bar{x}), \ with \quad \bar{x} = x \quad \ when \quad \ \tau = 0. \tag{1.23}$$

Example 1.21 *The group of translations in the x direction*

$\bar{x} = x + \varepsilon$ *and* $\bar{y} = y$*, where* $\varepsilon \in \mathbb{R}$*,*

is equivalent to the solution of an initial value problem

$\frac{d\bar{x}}{d\varepsilon} = 1, \frac{d\bar{y}}{d\varepsilon} = 0$ *with* $\bar{x} = x, \bar{y} = y$ *at* $\varepsilon = 0$*.*

Definition 1.24 ***(Infinitesimal generator)*** *The* ***infinitesimal generator*** *of the one-parameter Lie group of transformations* $\bar{x} = \varphi(x\ ;\varepsilon)$ *is the operator*

$$\mathbf{X} = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{1.24}$$

where $\xi_i = \frac{\partial \bar{x}_i}{\partial \varepsilon}\Big|_{\varepsilon=0}$.

Theorem 1.22 *The one-parameter Lie group of transformations* $\bar{x} = \varphi(x\ ;\varepsilon)$ *can* *be written as :*

$$\begin{aligned}\bar{x} &= e^{\varepsilon\mathbf{X}}x \\ &= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathbf{X}^k x,\end{aligned} \tag{1.25}$$

where the operator $\mathbf{X}$ *is the infinitesimal generator of the Lie group.*

Example 1.23 *Consider the rotation group:*

$$\bar{x} = x\cos\varepsilon + y\sin\varepsilon, \quad \bar{y} = -x\sin\varepsilon + y\cos\varepsilon \tag{1.26}$$

The infinitesimals $\xi(x,y) = \frac{\partial\bar{x}}{\partial\varepsilon}\big|_{\varepsilon=0} = y$ *and* $\eta(x,y) = \frac{\partial\bar{y}}{\partial\varepsilon}\big|_{\varepsilon=0} = -x$ *define the symmetry generator associated with (1.26) as*

$$\mathbf{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{1.27}$$

Alternatively, given the symmetry generator, one can find the transformation associated with that generator as shown below.

Consider the Taylor series corresponding to the generator given by

$$(\bar{x},\bar{y}) = (e^{\varepsilon\mathbf{X}}x, e^{\varepsilon\mathbf{X}}y), \tag{1.28}$$

where $\mathbf{X}x = y$*,* $\mathbf{X}^2 x = -x$ *and* $\mathbf{X}^3 x = -y$ *etc. Then*

$$\begin{aligned}
\bar{x} &= e^{\varepsilon\mathbf{X}}x \\
&= \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathbf{X}^k x \\
&= (1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \cdots)x + (\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} - \cdots)y \\
&= x\cos\varepsilon + y\sin\varepsilon. \qquad (1.29)
\end{aligned}$$

Similarly,

$$\bar{y} = e^{\varepsilon\mathbf{X}}y = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\mathbf{X}^k y = -x\sin\varepsilon + y\cos\varepsilon. \qquad (1.30)$$

Theorem 1.24 ***(Change of coordinates)*** *The infinitesimal symmetry generator*

$$\mathbf{X} = \sum_{i=1}^{n}\xi_i(\alpha)\frac{\partial}{\partial\alpha_i} = \sum_{i=1}^{n}X(\alpha_i)\frac{\partial}{\partial\alpha_i} \qquad (1.31)$$

in coordinates α_i *can be transformed to new coordinates* β_i *by the application of the infinitesimal symmetry generator to coordinates* β_i*, through the following formula:*

$$\mathbf{X} = \sum_{i=1}^{n}X(\beta_i)\frac{\partial}{\partial\beta_i}. \qquad (1.32)$$

Definition 1.25 ***(Canonical Coordinates)*** *A change of coordinates* $y = (y_1, y_2, ..., y_n)$ *defines a set of canonical coordinates for the one-parameter Lie group of transformations* $\bar{x} = \varphi(x\ ;\varepsilon)$ *if in terms of such coordinates the group* *becomes*

$$\bar{y_i} = y_i, \qquad i = 1, 2, ..., n-1,$$
$$\bar{y_n} = y_n + \varepsilon. \tag{1.33}$$

Theorem 1.25 *For any Lie group of transformations $\bar{x} = \varphi(x\ ;\varepsilon)$ there exists a set of canonical coordinates $y = (y_1, y_2, ..., y_n)$ that make the Lie group in the form (1.33).*

Theorem 1.26 *In terms of any set of canonical coordinates $y = (y_1, y_2, ..., y_n)$, the infinitesimal generator of the one-parameter Lie group of transformations $\bar{x} = \varphi(x\ ;\varepsilon)$ is $\boldsymbol{Y} = \frac{\partial}{\partial y_n}$.*

1.4.2 Infinitesimal criterion for the invariance of functions and surfaces

Definition 1.26 *Let $\bar{x} = \varphi(x\ ;\varepsilon)$ be the Lie group of transformations of one parameter ε and let $f(x)$ be an infinitely differentiable function.*

- *The function $f(x)$ is said to be an **invariant function** if and only if*

$$f(\bar{x}) = f(x) \tag{1.34}$$

- *The surface $f(x) = 0$ is said to be an **invariant surface** if and only if $f(\bar{x}) = 0$ when $f(x) = 0$.*

Theorem 1.27 *A function $f(x)$ is an invariant function of the Lie group of trans-*

formation $\bar{x} = \varphi(x\ ;\varepsilon)$ if and only if $\boldsymbol{X}f(x) \equiv 0$, where $\boldsymbol{X}$ is the infinitesimal generator of the transformation $\bar{x} = \varphi(x\ ;\varepsilon)$.

Theorem 1.28 *A surface $f(x) = 0$ is an invariant surface of the Lie group of transformation $\bar{x} = \varphi(x\ ;\varepsilon)$ if and only if $\boldsymbol{X}f(x) = 0$ when $f(x) = 0$ where $\boldsymbol{X}$ is the infinitesimal generator of the transformation $\bar{x} = \varphi(x\ ;\varepsilon)$.*

1.4.3 Point transformations and Prolongations

Consider the kth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^\alpha(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m\ , \tag{1.35}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second,..., k^{th} order partial derivatives,

i.e., $u^\alpha_{(i)} = D_i(u^\alpha), u^\alpha_{(i)(j)} = D_j D_i(u^\alpha)$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_{(i)} \frac{\partial}{\partial u^\alpha} + u^\alpha_{(i)(j)} \frac{\partial}{\partial u^\alpha_{(j)}} + ..., \quad i = 1, ..., n\ , \tag{1.36}$$

in which the summation convention is used.

To apply the infinitesimal criterion for the invariance of the kth-order system of partial differential equations of n independent variables and m dependent variables, one needs to extend the infinitesimal generator to include all the dependent variables and their derivatives. In this section we discuss the prolongation formula for a kth-order system of PDE which consists of n independent and m dependent variables.

Definition 1.27 *A one-parameter Lie group of point transformations is a group of transformations of the form*

$$\begin{aligned} \bar{x} &= \varphi(x, u\ ; \varepsilon) \\ \bar{u} &= \psi(x, u\ ; \varepsilon) \end{aligned} \tag{1.37}$$

acting on the space of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$.

Definition 1.28 *The k^{th} prolongation of the one-parameter Lie group of point transformations (1.37) is the following one-parameter Lie group of transformations acting on $(x, u, u_{(1)}, ..., u_{(k)})$-space:*

$$\begin{aligned} &\bar{x}_i = \varphi(x, u\ ; \varepsilon) = x_i + \varepsilon\ \xi^i(x, u) + O(\varepsilon^2) \\ &\bar{u}^\alpha = \psi^\alpha(x, u\ ; \varepsilon) = u^\alpha + \varepsilon\ \eta^\alpha(x, u) + O(\varepsilon^2) \\ &\bar{u}^\alpha_{(i)} = \chi^\alpha(x, u, u_{(1)}\ ; \varepsilon) = u^\alpha_{(i)} + \varepsilon\ \zeta^\alpha_i(x, u, u_{(1)}) + O(\varepsilon^2) \\ &\vdots \\ &\bar{u}^\alpha_{(i_1)\ldots(i_s)} = \chi^\alpha_{i_1\ldots i_s}(x, u, \ldots, u_{(k)}\ ; \varepsilon) = u^\alpha_{(i_1)\ldots(i_s)} + \varepsilon\ \zeta^\alpha_{i_1\ldots i_s}(x, u, \ldots, u_{(k)}) + O(\varepsilon^2) \end{aligned} \tag{1.38}$$

with the corresponding k^{th} prolongation infinitesimal generator

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 \ldots i_s} \frac{\partial}{\partial u^\alpha_{(i_1)\ldots(i_s)}}, \tag{1.39}$$

where the coefficients $\zeta^\alpha_{i_1\ldots i_s}$ are determined uniquely by the prolongation formulae [8] as :

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{(i)(j)} \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{(j)(i_1)\ldots(i_s)}, \quad s > 1, \end{aligned} \tag{1.40}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_{(j)}. \tag{1.41}$$

Theorem 1.29 ***(Infinitesimal criterion for the invariance of PDE)*** *The surfaces $E^\alpha(x, u, u_{(1)}, ..., u_{(k)}) = 0, \alpha = 1, ..., m$ are invariant surfaces under the one-parameter Lie group of point transformations (1.38) if and only if $\boldsymbol{X}E^\alpha = 0$ when $E^\alpha = 0,\ \alpha = 1, ..., m$, where $\boldsymbol{X}$ is the kth-extended infinitesimal generator of the point transformation (1.39).*

Definition 1.29 *The one-parameter Lie group of point transformations (1.38) leaving the surfaces $E^\alpha(x, u, u_{(1)}, ..., u_{(k)}) = 0, \alpha = 1, ..., m$ invariant is called the symmetry group of the system of PDEs (1.35).*

Definition 1.30 *$u = \Theta(x)$ with component $u^\alpha = \Theta^\alpha(x), \alpha = 1, \ldots, m$ is an invariant solution of the system of PDEs (1.35) resulting from an admitted point* *symmetry with infinitesimal generator*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{1.42}$$

if and only if

- $u^\alpha = \Theta^\alpha(x)$ *is an invariant surface of (1.42) for each* $\alpha = 1, \ldots, m$*.*
- $u = \Theta(x)$ *solves (1.35).*

The invariant solutions can be obtained through symmetry reductions carried out by implementing the well-known procedure of utilizing the joint invariants of the subalgebras obtained by three-dimensional optimal systems, see, e.g. [8, 13, 19] for details.

Remark 1.30 *[18] Let* $\langle X_1, \ldots, X_n\rangle$ *be a Lie algebra with basis*

$$X_i = \xi_i^1 \frac{\partial}{\partial t} + \xi_i^2 \frac{\partial}{\partial r} + \xi_i^3 \frac{\partial}{\partial \theta} + \xi_i^4 \frac{\partial}{\partial \varphi} + \eta_i \frac{\partial}{\partial u}, i = 1, \ldots, n.$$

A necessary condition for the existence of invariant solution under the Lie algebra $\langle X_1, \ldots, X_n\rangle$ *is the following* ***transversality condition****:*

$$rank\{E_1\} = rank\{E_2\} \tag{1.43}$$

where

$$E_1 = \begin{pmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 & \xi_1^4 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & \xi_n^3 & \xi_n^4 \end{pmatrix}, \ E_2 = \begin{pmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 & \xi_1^4 & \eta_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & \xi_n^3 & \xi_n^4 & \eta_n \end{pmatrix}.$$

1.4.4 Equivalence of invariant solutions

Before giving the formal definition of equivalent invariant solutions, let's note the following general fact:

Whenever a transformation of a one-parameter Lie group G operates on each element in a set S and U is a subset of S and H is the stabilizer of U, where by stabilizer we mean the set of all transformations in G that fix U, the stabilizer of $a.U,\ a \in G$ is aHa^{-1}. We will apply this where the set S is the set of solutions of a differential equation, U is the set of invariant solutions and the group G is the local group whose Lie algebra is the symmetry algebra of the differential equation.

Definition 1.31 *Consider the differential equation (1.35) admitting the group of transformations G. Let $\mathcal{L}$ be the Lie algebra corresponding the group G. If $u = \Theta_1(x)$ and $u = \Theta_2(x)$ are two invariant solutions of the given differential equation under the subalgebras H_1 and H_2 of $\mathcal{L}$, respectively, then we call $u = \Theta_1(x)$ and $u = \Theta_2(x)$ **equivalent invariant solutions** with respect to the group G if one can find some transformation in G that transforms $u = \Theta_1(x)$ to $u = \Theta_2(x)$.*

Let H_1 be conjugate to H_2 with respect to the group of transformations G. Define U to be the set of invariant surfaces under H_1. Then H_1 belongs to the stabilizer of U and H_2 belongs to the stabilizer of $a.U$ for some $a \in G$. The set of invariant surfaces under H_2 should be of the form $a.U$.

Therefore, the problem of classifying the invariant solutions is reduced to classifying the corresponding conjugacy classes of subalgebras of the symmetry algebra $\mathcal{L}$ [19].

1.5 Wave equations

The wave equation on a spacetime is given by $\Delta_g u = 0$ where $\Delta_g = \frac{\partial}{\partial x_i}(\sqrt{\mid g \mid} g^{ik} \frac{\partial}{\partial x_k})$ is called the Laplace-Beltrami operator for the metric given by

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - e^{\mu(r,t)} d\theta^2 - e^{\mu(r,t)} \sin^2\theta d\varphi^2. \tag{1.1}$$

Hence, the wave equation $\Delta_g u = 0$ on the metric (1.1) can be written as

$$\frac{\partial}{\partial t}\left(e^{\left(\mu - \frac{\nu}{2} + \frac{\lambda}{2}\right)} \sin\theta \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial r}\left(e^{\left(\mu + \frac{\nu}{2} - \frac{\lambda}{2}\right)} \sin\theta \frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(e^{\left(\frac{\nu}{2} + \frac{\lambda}{2}\right)} \sin\theta \frac{\partial u}{\partial \theta}\right) - \frac{\partial}{\partial \varphi}\left(\frac{e^{\left(\frac{\nu}{2} + \frac{\lambda}{2}\right)}}{\sin\theta} \frac{\partial u}{\partial \varphi}\right) = 0. \tag{1.2}$$

The procedure of obtaining the Lie point symmetries of wave equation (1.2) is summarized below. Take the one-parameter Lie group of point transformations in $(t, r, \theta, \varphi, u)$ given by

$$\begin{aligned} t^* &= t + \varepsilon\xi^1 + O(\varepsilon^2) \\ r^* &= r + \varepsilon\xi^2 + O(\varepsilon^2) \\ \theta^* &= \theta + \varepsilon\xi^3 + O(\varepsilon^2) \\ \varphi^* &= \varphi + \varepsilon\xi^4 + O(\varepsilon^2) \\ u^* &= u + \varepsilon\eta + O(\varepsilon^2) \end{aligned} \tag{1.3}$$

as $\varepsilon \to 0$, where ξ^i, η are functions of $(t, r, \theta, \varphi, u)$ and ε is the group parameter. Therefore, the form of the corresponding generator becomes

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial \theta} + \xi^4 \frac{\partial}{\partial \varphi} + \eta \frac{\partial}{\partial u}. \tag{1.4}$$

In the next chapters, we use the notation $X = [\xi^1,\, \xi^2,\, \xi^3,\, \xi^4, \eta]$ to express the vector field X and a similar notation will be used for Cartesian coordinates (t, x, y, z, u).

Let us denote by $X^{[2]}$ the second prolongation of X. Then using the invariance criterion given by Theorem 1.29 gives the determining equations for $\xi^1,\, \xi^2,\, \xi^3,\, \xi^4, \eta$. The determining equations constitute a system of linear homogeneous PDEs for $\xi^1,\, \xi^2,\, \xi^3,\, \xi^4, \eta$ with $t, r,\, \theta, \varphi,\, u$ as independent variables. Solving this system provides the explicit form of the infinitesimals $\xi^1,\, \xi^2,\, \xi^3,\, \xi^4, \eta$ of the symmetry generators. The obtained symmetry generators of the form (1.4) form a Lie algebra and the corresponding symmetry transformations of the form (1.3) form a one-parameter Lie group. The one-parameter Lie group of the symmetry transformations can be derived from the generators (1.4) by solving the initial value problems:

$$\begin{aligned} &\tfrac{d\tilde{x}_i(\varepsilon)}{d\varepsilon} = \xi^i\left(\tilde{x}(\varepsilon), \tilde{u}(\varepsilon)\right), \ \tfrac{d\tilde{u}(\varepsilon)}{d\varepsilon} = \eta\left(\tilde{x}(\varepsilon), \tilde{u}(\varepsilon)\right), \\ &\tilde{x}_i(0) = x_i, \ \tilde{u}(0) = u, \ i = 1, ..., 4. \end{aligned} \tag{1.5}$$

where $x = (t, r, \theta, \varphi)$.

Another approach to find the symmetries of the wave equation using the conformal Killing vector field of the underlying spacetimes metric is explained in the next theorem.

Theorem 1.31 *[10] Let M^n be a Lorentzian manifold of dimension $n \geq 3$ with the metric g given in local coordinates $\{x_1, x_2, ..., x_n\}$. The Lie symmetries of wave equation $\Delta_g u = 0$ on M^n have the form*

$$X = \xi^i(x)\frac{\partial}{\partial x_i} + \left(\left(\frac{2-n}{4}\mu(x) + c\right)u + b(x)\right)\frac{\partial}{\partial u} \tag{1.6}$$

where c is an arbitrary constant,

$$\Delta_g b(x) = 0, \quad \Delta_g \mu(x) = 0, \tag{1.7}$$

$Y = \xi^i(x)\frac{\partial}{\partial x_i}$ is a conformal Killing vector field of the metric g such that

$$(\pounds_Y g)_{ab} = \xi^c\partial_c g_{ab} + g_{cb}\partial_a\xi^c + g_{ca}\partial_b\xi^c = \mu(x)g_{ab} \tag{1.8}$$

and $\pounds_Y$ *denotes the Lie derivative with respect to vector field* Y*, where* $b(x)$ *and* $\mu(x)$ *satisfy* (1.7).

CHAPTER 2

WAVE EQUATION ON EINSTEIN SPACETIME

In this chapter, we will discuss the wave equation $\Delta u = 0$ on the spherically symmetric space time with the metric

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{\mu(t,r)}d\theta^2 - e^{\mu(t,r)}\sin^2\theta d\varphi^2 \tag{2.1}$$

where $\nu = 0$, $\lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2$, $\alpha = -c^2 < 0$. Hence the corresponding wave equation on this metric is

$$\begin{aligned}&\tfrac{\partial}{\partial t}\left(\tfrac{r^2 u_t \sin\theta}{\sqrt{1-c^2r^2}}\right) - \tfrac{\partial}{\partial r}(r^2 u_r \sin\theta\sqrt{1-c^2r^2}) - \tfrac{\partial}{\partial \theta}\left(\tfrac{u_\theta \sin\theta}{\sqrt{1-c^2r^2}}\right) - \\ &\tfrac{\partial}{\partial \varphi}\left(\tfrac{u_\varphi}{\sin\theta\sqrt{1-c^2r^2}}\right) = 0.\end{aligned} \tag{2.2}$$

In fact, finding the symmetry transformations, joint invariants and reductions in spherical coordinates has a lot of difficulties. Therefore, we change to Cartesian coordinates.

Equation (2.2) can be written in Cartesian coordinates through the transformation $x = r\cos\varphi\sin\theta, y = r\sin\varphi\sin\theta, z = r\cos\theta$ as:

$$\begin{aligned}&(c^2x^2-1)\,u_{xx} + (c^2y^2-1)\,u_{yy} + (c^2z^2-1)\,u_{zz} + 3\,c^2zu_z + 2\,c^2xzu_{zx}\\&+3\,c^2xu_x + 2\,c^2xyu_{xy} + 2\,c^2yzu_{yz} + 3\,c^2yu_y + u_{tt} = 0\end{aligned} \tag{2.3}$$

2.1 Lie point symmetries of the wave equation on Einstein spacetime

By using Theorem 1.31 and the isometries of the metric (2.1) given in [21], the Lie symmetry algebra of the wave equation (2.2) consists of the eight-dimensional subalgebra spanned by

$$\begin{aligned}X_1 &= [0, A\sin\varphi\sin\theta, \tfrac{A\sin\varphi\cos\theta}{r}, \tfrac{A\cos\varphi}{r\sin\theta}, 0]\\X_2 &= [0, A\cos\varphi\sin\theta, \tfrac{A\cos\varphi\cos\theta}{r}, -\tfrac{A\sin\varphi}{r\sin\theta}, 0]\\X_3 &= [0, A\cos\theta, -\tfrac{A\sin\theta}{r}, 0, 0]\\X_4 &= [0, 0, -\cos\varphi, \tfrac{\sin\varphi}{\tan\theta}, 0]\\X_5 &= [0, 0, \sin\varphi, \tfrac{\cos\varphi}{\tan\theta}, 0]\\X_6 &= [0, 0, 0, 1, 0]\\X_7 &= [1, 0, 0, 0, 0]\\X_8 &= [0, 0, 0, 0, u]\end{aligned} \tag{2.4}$$

and the infinite-dimensional ideal consisting of the operators

$$X_\tau = \tau(t, r, \theta, \varphi)\frac{\partial}{\partial u} \tag{2.5}$$

where $\tau(t, r, \theta, \varphi)$ is an arbitrary solution of the wave equation (2.2) and $A = \sqrt{1 - c^2 r^2}$.

Moreover, the generators (2.4) can be written in Cartesian coordinates as follows:

$$\begin{aligned}
X_1 &= [0, 0, B, 0, 0] \\
X_2 &= [0, B, 0, 0, 0] \\
X_3 &= [0, 0, 0, B, 0] \\
X_4 &= [0, -z, 0, x, 0] \\
X_5 &= [0, 0, z, -y, 0] \\
X_6 &= [0, -y, x, 0, 0] \\
X_7 &= [1, 0, 0, 0, 0] \\
X_8 &= [0, 0, 0, 0, u].
\end{aligned} \tag{2.6}$$

where $B = \sqrt{1 - c^2(x^2 + y^2 + z^2)}$.

2.2 Lie point symmetry transformations

The one-parameter groups $G_i(\varepsilon) = \{e^{\varepsilon X_i} : \varepsilon \in \mathbb{R}\}$ generated by (2.6) are given by solving the initial value problems (1.5) as follows:

$$
\begin{aligned}
&G_1(\varepsilon_1) : (t,x,y,z,u) \mapsto (t, x, \tfrac{1}{c}\sqrt{1-c^2(\,x^2+\,z^2)}\,\sin\left(\arctan\left(\tfrac{c\;y}{B}\right)+c\;\,\varepsilon_1\right), z, u)\\
&G_2(\varepsilon_2) : (t,x,y,z,u) \mapsto (t, \tfrac{1}{c}\sqrt{1-c^2(\,y^2+\,z^2)}\,\sin\left(\arctan\left(\tfrac{c\;x}{B}\right)+c\;\,\varepsilon_2\right), y, z, u)\\
&G_3(\varepsilon_3) : (t,x,y,z,u) \mapsto (t, x, y, \tfrac{1}{c}\sqrt{1-c^2(\,x^2+\,y^2)}\,\sin\left(\arctan\left(\tfrac{c\;z}{B}\right)+c\;\,\varepsilon_3\right), u)\\
&G_4(\varepsilon_4) : (t,x,y,z,u) \mapsto (t, x\sin\varepsilon_4 - z\cos\varepsilon_4, y, x\sin\varepsilon_4 + z\cos\varepsilon_4, u)\\
&G_5(\varepsilon_5) : (t,x,y,z,u) \mapsto (t, x, -z\sin\varepsilon_5 - y\cos\varepsilon_5, y\sin\varepsilon_5 - z\cos\varepsilon_5, u)\\
&G_6(\varepsilon_6) : (t,x,y,z,u) \mapsto (t, -y\sin\varepsilon_6 + x\cos\varepsilon_5, x\sin\varepsilon_6 + y\cos\varepsilon_6, z, u)\\
&G_7(\varepsilon_7) : (t,x,y,z,u) \mapsto (t+\varepsilon_7, y, z, u)\\
&G_8(\varepsilon_8) : (t,x,y,z,u) \mapsto (t, x, y, z, u+\varepsilon_8)
\end{aligned}
\tag{2.7}
$$

2.3 Lie Algebra Structure

The non-zero Lie brackets of (2.4) are:

$$
\begin{array}{llll}
[X_1,X_2] = c^2X_6, & [X_1,X_3] = c^2X_5, & [X_1,X_5] = -X_3, & [X_1,X_6] = -X_2,\\
[X_2,X_3] = -c^2X_4, & [X_2,X_4] = X_3, & [X_2,X_6] = X_1, & [X_3,X_4] = -X_2,\\
[X_3,X_5] = X_1, & [X_4,X_5] = X_6, & [X_4,X_6] = -X_5, & [X_5,X_6] = X_4.
\end{array}
\tag{2.8}
$$

The Levi-Decomposition of this algebra is $\mathcal{L} = \{X_1, X_2, X_3, X_4, X_5, X_6\} \oplus \{X_7, X_8\}$. Let S be the semisimple part. To identify the semisimple part, we need to find a Cartan algebra and the corresponding root space decomposition.

First of all, after computing the Killing form, we see that it is negative definite. Thus to determine a Cartan algebra, choose any non-zero element in the semisimple part S. We choose, for example, the element X_3 and compute its centralizer. The centralizer turns out to be $\{X_3, X_6\}$ and the subalgebra $\{X_3, X_6\}$ is self centralizing. Thus $C = \{X_3, X_6\}$ is a Cartan subalgebra which is itself the only maximal solvable subalgebra up to the conjugacy as mentioned in section 1.3.1.3. The roots for this Cartan subalgebra are $\{(ci, i), (-ci, i), (-ci, -i), (ci, -i)\}, i = \sqrt{-1}$. Therefore, the positive roots are $\{(ci, i), (ci, -i)\}$. The root vectors for the positive roots are $\{X_1 + cX_4 + i(X_2 + cX_5), X_1 - cX_4 + i(X_2 - cX_5)\}$. Since the negative roots are conjugates of the positive roots, the real and the imaginary parts of the positive root vectors must generate, as a Lie algebra, the full Lie algebra.

This gives us the change of basis which makes the computations easier:

$$\begin{aligned} &V_1 = X_1 + cX_4, \quad V_2 = X_2 + cX_5, \quad V_3 = X_3 - cX_6, \quad V_4 = X_1 - cX_4, \\ &V_5 = X_2 - cX_5, \quad V_6 = X_3 + cX_6, \quad V_7 = X_7, \qquad\qquad V_8 = X_8. \end{aligned} \tag{2.9}$$

The corresponding Lie brackets of this subalgebra are:

$$\begin{gathered} [V_1, V_2] = -2cV_3 \quad [V_1, V_3] = 2cV_2, \quad [V_2, V_3] = -2cV_1, \quad [V_4, V_5] = 2cV_6, \\ [V_4, V_6] = -2cV_5, \quad [V_5, V_6] = 2cV_4. \end{gathered} \tag{2.10}$$

It is obvious from (2.10) that the subalgebra $\langle V_1, V_4 \rangle$ is Cartan since it is abelian and it is self centralizing. Since our Lie algebra is compact, therefore, $\langle V_1, V_4 \rangle$ is the only maximal solvable algebra up to the conjugacy. The subalgebra $\langle V_1, V_4\rangle$ is conjugate to $\langle V_3, V_6\rangle$.

As we will see later, $\langle V_1, V_2, V_3\rangle$ and $\langle V_4, V_5, V_6\rangle$ form two copies of $so(3)$ which commute with each other. Since $so(4)$, which is the set of all skew symmetric 4×4 matrices, is also isomorphic to $so(3)\oplus so(3)$ [22], we see that the semisimple part is isomorphic to $so(4)$. This decomposition can be obtained by working with a Cartan subalgebra of $so(4)$ and determine its root space decomposition as was done above.

2.4 Optimal systems of solvable subalgebras of $so(4)$

To find the optimal systems of $so(4)$, we first find the one-dimensional optimal system and a rough classification of higher order subalgebras inside the maximal solvable subalgebra $\langle V_1, V_4\rangle$ which is abelian. Secondly, we obtain the higher-dimensional optimal system by removing the repetitions from the obtained rough classification of subalgebras using the adjoint representation of $so(4)$.

Theorem 2.1 *The optimal systems Θ_i up to order three of solvable subalgebras of $so(4)$ with the non-zero Lie brackets (2.10) are the following:*

- *The 1-dimensional optimal system Θ_1 is $\{\langle V_4\rangle\}\cup\{\langle V_1+\alpha V_4\rangle : \alpha\in\mathbb{R}\}$,*
- *The 2-dimensional optimal system Θ_2 is $\{\langle V_1, V_4\rangle\}$.*

There is no 3-dimensional optimal system.

Proof. Clearly, it is enough to deal with the one-dimensional optimal system only. The one-dimensional optimal system of the maximal solvable subalgebra of $so(4)$ is itself the one-dimensional optimal system of $so(4)$. This is because the representative elements are non-conjugate under the adjoint representation of $so(4)$ given by

$$A(\varepsilon_1, \ldots, \varepsilon_6) = A_1(\varepsilon_1) \ldots A_6(\varepsilon_6), \tag{2.11}$$

where $A_i(\varepsilon_i)$ are given in Remark 1.10. ∎

2.5 Optimal systems of solvable subalgebras of $\mathcal{L} = so(4) \oplus \mathbb{R}^2$

First, note the general fact that if $\mathcal{L} = S \oplus R$ where S is the semisimple part and the radical R is the center, then the conjugacy classes of S can be joined with elements of the center to obtain conjugacy classes of $\mathcal{L}$, as follows

Let $\pi : S \oplus R \to S$ be the projection defined by $\pi(x, y) = x$. This is a homomorphism because R is an ideal. Therefore, it will map conjugate classes to conjugate classes.

Every k-dimensional subalgebra of $\mathcal{L}$ is of the form $\langle x_1 + y_1, x_2 + y_2, ..., x_k + y_k\rangle$ where $x_i \in S,\ y_i \in R$. Its projection is $\langle x_1, ..., x_k\rangle$ of dimension less than or equal to k. Moreover, if $\langle x_1 + y_1, x_2 + y_2, ..., x_k + y_k\rangle$ is conjugate to $\langle \tilde{x_1} + \tilde{y_1}, \tilde{x_2} + \tilde{y_2}, ..., \tilde{x_k} + \tilde{y_k}\rangle$, then as the radical R is the center, $y_i = \tilde{y_i}$ and $\langle x_1, x_2, ..., x_k\rangle$ is conjugate to $\langle \tilde{x_1}, \tilde{x_2}, ..., \tilde{x_k}\rangle$. However, the dimension of the image algebra of

$\langle \tilde{x_1} + \tilde{y_1}, \tilde{x_2} + \tilde{y_2}, ..., \tilde{x_k} + \tilde{y_k} \rangle$ can go down. Thus to get all conjugacy classes of the full algebra, we start with the elements of the optimal systems of S and add to each one of them arbitrary elements of the center and keep those that form a subalgebra. The classes of the center correspond to the zero subspace of S. This will give all the conjugacy classes of the full algebra. Applying this to $\mathcal{L} = so(4) \oplus \mathbb{R}^2$, we obtain the following classes.

Clearly, the 1-dimensional optimal system $\tilde{\Theta}_1$ of $\mathbb{R}^2$ is $\{\langle V_8 \rangle\} \cup \{\langle V_7 + \alpha V_8 \rangle : \alpha \in \mathbb{R}\}$ and the only 2-dimensional optimal system $\tilde{\Theta}_2$ of $\mathbb{R}^2$ is $\{\langle V_7, V_8 \rangle\}$.

In order to get the optimal systems of the full Lie algebra up to order three, we use the optimal systems of $so(4)$ constructed in Theorem 2.1. We join it with the optimal system of the abelian algebra $\mathbb{R}^2$ as explained above.

- To get the one-dimensional optimal system of $\mathcal{L}$, we have the cases:

  i) We add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_1, in this case we get $\{\langle V_4 + Z_1 \rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1 \rangle : \alpha \in \mathbb{R}\}$.

  ii) We take $\tilde{\Theta}_1$ itself, in this case we get $\{\langle V_8 \rangle\} \cup \{\langle V_7 + \beta V_8 \rangle : \beta \in \mathbb{R}\}$.

- To get the two-dimensional optimal system of $\mathcal{L}$, we have the cases:

  i) We add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_2, in this case we get $\{\langle V_1 + Z_1, V_4 + Z_2 \rangle\}$.

  ii) We add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_1 and combine the result with an element from $\tilde{\Theta}_1$, in this case we get $\{\langle V_4 + Z_1, Z_3 \rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1, Z_3 \rangle : \alpha \in \mathbb{R}\}$.

iii) Take $\tilde{\Theta}_2$ itself, in this case we get $\{\langle V_7, V_8\rangle\}$.

- To get the three-dimensional optimal system of $\mathcal{L}$,

  i) either we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_2 and combine the result with an element from $\tilde{\Theta}_1$, in this case we get $\{\langle V_1 + Z_1, V_4 + Z_2, Z_3\rangle\}$;

  ii) or we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_1 and combine the result with an element from $\tilde{\Theta}_2$, in this case we get $\{\langle V_4 + Z_1, V_7, V_8\rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1, V_7, V_8\rangle : \alpha \in \mathbb{R}\}$.

Finally, we check that the obtained class is a subalgebra by taking the wedge product of it with its commutator and equate by zero and see if we can kill some constants. This leads to the following theorem

Theorem 2.2 *The optimal systems of solvable subalgebra of $\mathcal{L}$ are as follows:*

- *The* 1*-dimensional optimal system is* $\{\langle V_4 + Z_1\rangle,\ \langle V_8\rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1\rangle,\ \langle V_7 + \beta V_8\rangle : \alpha, \beta \in \mathbb{R}\}$.

- *The* 2*-dimensional optimal system is*

  $\{\langle V_1 + Z_1, V_4 + Z_2\rangle, \langle V_4 + Z_1, Z_3\rangle, \langle V_7, V_8\rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1, Z_3\rangle : \alpha \in \mathbb{R}\}$.

- *The* 3*-dimensional optimal system is*

  $\{\langle V_1 + Z_1, V_4 + Z_2, Z_3\rangle, \langle V_4 + Z_1, V_7, V_8\rangle\} \cup \{\langle V_1 + \alpha V_4 + Z_1, V_7, V_8\rangle : \alpha \in \mathbb{R}\}$.

Here, $Z_1 = \alpha_1 V_7 + \beta_1 V_8$, $Z_2 = \alpha_2 V_7 + \beta_2 V_8$ *are arbitrary elements of* $\mathbb{R}^2$ *and* $Z_3 = V_7 + \alpha_3 V_8$ *or* $Z_3 = V_8$ *represents a one-dimensional optimal system of* $\mathbb{R}^2$.

2.6 Three-dimesional optimal system of non-solvable subalgebras of $\mathcal{L} = so(4) \oplus \mathbb{R}^2$

If H is a three-dimensional non-solvable algebra, then H equals its commutator. As the commutator of $\mathcal{L}$ is $so(4)$, all such subalgebras of $\mathcal{L}$ are subalgebras of $so(4)$. To find such subalgebras we follow the algorithm given in section 1.3.1.3.

We need to construct the copies of $so(3)$ and $sl(2,\mathbb{R})$, if any, by following the algorithm given in section 1.3.1.3:

- First, construct the copies of $so(3)$:

  i) The element V_1 has the eigenvector V_2+iV_3 corresponding to the eigenvalue $2ci$. Therefore, $\tilde{V_1} = \frac{V_1}{2c}$ has the same eigenvector with the eigenvalue i. Moreover, $[V_2, V_3] = -(2c)^2\tilde{V_1}$. Hence, $\langle V_1, V_2, V_3\rangle$ forms a copy of $so(3)$.

  ii) The element V_4 has the eigenvector V_5+iV_6 corresponding to the eigenvalue $-2ci$. Therefore, $\tilde{V_4} = \frac{V_4}{-2c}$ has the same eigenvector with the eigenvalue i. Moreover, $[V_5, V_6] = -(2c)^2\tilde{V_4}$. Hence, $\langle V_4, V_5, V_6\rangle$ forms a copy of $so(3)$.

  iii) The element $V_1+\alpha V_4$ has the eigenvector $V_2+V_5+i(V_3-V_6)$ corresponding to the eigenvalue $2ci$. Therefore, $\tilde{V} = \frac{V_1+\alpha V_4}{2c}$ has the same eigenvector with the eigenvalue i. Moreover, and $[V_2+V_5, V_3-V_6] = -2c\tilde{V}$. Hence, $\langle V_1+V_4, V_2+V_5, V_3-V_6\rangle$ forms a copy of $so(3)$. Note that here α must be equal to one to ensure that $\langle V_1+\alpha V_4, V_2+V_5, V_3-V_6\rangle$ is a subalgebra.

- The Lie algebra $so(4)$ does not contain any copy of $sl(2,\mathbb{R})$, since it does not contain any non-abelian two-dimensional subalgebra.

This proves the following theorem

Theorem 2.3 *The* 3*-dimensional optimal system of non-solvable subalgebras of* $\mathcal{L}$ *is*

$\{\langle V_1, V_2, V_3\rangle, \langle V_4, V_5, V_6\rangle, \langle V_1+V_4, V_2+V_5, V_3-V_6\rangle\}$*, where the* $V_i, i = 1, \ldots, 6$ *form a basis of* $so(4)$ *given in (2.9).*

2.7 Joint invariants and invariant solutions of three-dimensional optimal systems of $\mathcal{L} = so(4) \oplus \mathbb{R}^2$

In this section, we compute the invariant solutions corresponding to three-dimensional subalgebras of $\mathcal{L}$.

2.7.1 Solvable subalgebras of $\mathcal{L}$

2.7.1.1 Case $\mathcal{L}_1 = \langle V_1 + Z_1, V_4 + Z_2, Z_3 \rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_1$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_1 + Z_1 &= [\alpha_1, -cz, \sqrt{1 - c^2(x^2 + y^2 + z^2)}, cx, \beta_1 u], \\ V_4 + Z_2 &= [\alpha_2, cz, \sqrt{1 - c^2(x^2 + y^2 + z^2)}, -cx, \beta_2 u], \\ Z_3 &= [1, 0, 0, 0, \alpha_3 u] \end{aligned} \tag{2.12}$$

The transversality condition (1.43) of (2.12) with rank three is always satisfied. Since the Lie algebra $\mathcal{L}_1$ is abelian, one can find the invariant functions, we call them also invariants, of $\mathcal{L}_1$ in any order. The invariants of Z_3 are:

$$m_1 = x, m_2 = y, m_3 = z, m_4 = u\mathrm{e}^{-\alpha_3 t} \tag{2.13}$$

The remaining operators can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$\begin{aligned} V_1 + Z_1 &= [-cm_3, \sqrt{1 - c^2({m_1}^2 + {m_2}^2 + {m_3}^2)}, cm_1, -m_4\left(\alpha_3\, \alpha_1 - \beta_1\right)], \\ V_4 + Z_2 &= [cm_3, \sqrt{1 - c^2({m_1}^2 + {m_2}^2 + {m_3}^2)}, -cm_1, -m_4\left(\alpha_3\, \alpha_2 - \beta_2\right)]. \end{aligned} \tag{2.14}$$

Next, the invariants of $V_1 + Z_1$ are

$$\begin{aligned} &n_1 = {m_1}^2 + {m_3}^2,\ n_2 = \arctan\left(\frac{c\ m_2}{\sqrt{1 - c^2({m_1}^2 + {m_2}^2 + {m_3}^2)}}\right) - \arctan\left(\frac{m_3}{m_1}\right), \\ &n_3 = m_4\ e^{\frac{(\alpha_3\, \alpha_1 - \beta_1)}{c} \arctan\left(\frac{m_3}{m_1}\right)} \end{aligned} \tag{2.15}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_4 + Z_2 = [0, -2\,c, -n_3\,((\alpha_1 + \alpha_2)\,\alpha_3 - \beta_2 - \beta_1)] \tag{2.16}$$

Finally, the invariants of $V_4 + Z_2$ are

$$n_1, n_3\ e^{\frac{(\beta_1 + \beta_2 - \alpha_3\,\alpha_1 - \alpha_3\,\alpha_2)}{2c} n_2} \tag{2.17}$$

Writing the invariants (2.17) in terms of the original variables gives the joint invariants of $\mathcal{L}_1$ as

$$x^2 + z^2,\ \ u\ e^{\left(A_1 \arctan\left(\frac{c\ y}{\sqrt{1-c^2(x^2+y^2+z^2)}}\right) + A_2 \arctan\left(\frac{z}{x}\right) - \alpha_3 t\right)} \tag{2.18}$$

where $A_1 = \frac{(\alpha_1+\alpha_2)\alpha_3 - \beta_1 - \beta_2}{2c}$, $A_2 = \frac{(3\alpha_1+\alpha_2)\alpha_3 - 3\beta_1 - \beta_2}{2c}$.

Therefore, the invariant transformations are:

$$w = x^2 + z^2, Z(w) =\ u\ e^{\left(A_1 \arctan\left(\frac{c\ y}{\sqrt{1-c^2(x^2+y^2+z^2)}}\right) + A_2 \arctan\left(\frac{z}{x}\right) - \alpha_3 t\right)} \tag{2.19}$$

Thus, using (2.19), equation (2.3) can be reduced to the ODE:

$$\begin{gathered} 4\Big(w^2\left(c^2 w - 1\right)^2\Big) Z'' + 4\Big(2c^4 w^3 - 3c^2 w^2 + w\Big) Z' \\ + \Big(\alpha^2 c^2 w^2 + \left(\left(A_1^2 - A_2^2\right) c^2 - \alpha^2\right) w + A_2^2\Big) Z = 0 \end{gathered} \tag{2.20}$$

It was found using Maple software that the transformation

$$w = \frac{r}{c^2},\ Z(w) = (r-1)^{-\frac{1}{2}\,iA_1}\, r^{\frac{1}{2}\,iA_2} R(r) \tag{2.21}$$

reduces equation (2.20) to the hypergeometric differential equation

$$r(r-1)R'' + \big((\nu+\mu+1)r - \gamma\big)R' + \nu\mu R = 0 \tag{2.22}$$

with $\nu = \frac{(1+iA_2-iA_1)c-\sqrt{c^2-\alpha_3{}^2}}{2c}, \mu = \frac{(1+iA_2-iA_1)c+\sqrt{c^2-\alpha_3{}^2}}{2c}, \gamma = 1 + iA_2$. The solution of (2.22) is given in terms of the hypergeometric function $F(\nu, \mu; \gamma; r)$ as

$$R(r) = c_1 F(\nu, \mu; \gamma; r) + c_2 r^{1-\gamma} F(\nu - \gamma + 1, \mu - \gamma + 1; 2 - \gamma; r) \tag{2.23}$$

Therefore, the solution of (2.20) is

$$Z(w) = (c^2 w - 1)^{\frac{1}{2}\,(\nu+\mu-\gamma)}\,(c^2 w)^{\frac{1}{2}\,(\gamma-1)} R(c^2 w) \tag{2.24}$$

Thus, the invariant solution of (2.3) is

$$u(t,x,y,z) = Z(x^2+z^2)e^{\alpha_3 t}\ e^{i\left((\gamma-1)\arctan\left(\frac{z}{x}\right)-(\nu+\mu-\gamma)\arctan\left(\frac{c\ \ y}{\sqrt{1-c^2(x^2+y^2+z^2)}}\right)\right)} \tag{2.25}$$

where $\alpha_3^2 = c^2 - c^2(\mu - \nu)^2$.

2.7.2 Non-solvable subalgebras of $\mathcal{L}$

As is well known, all three-dimensional non-solvable subalgebras are simple. As they have no non-trivial ideal, we use the method of reduced row echelon form of operators in any convenient basis. As shown in [5], the operators of the three-dimensional non-solvable subalgebra in the reduced row echelon form always form an abelian algebra. Clearly, the joint invariants of the three-dimensional non-solvable subalgebra are the same as those of this abelian algebra. Using this we find that the joint invariants for $\mathcal{L}$ as follows:

2.7.2.1 Case $\mathcal{L}_1 = \langle V_1, V_2, V_3 \rangle$

By writing $\mathcal{L}_1$ in the reduced row echelon form, the fundamental set of the invariants can be obtained by solving the following system

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} I_t \\ I_x \\ I_y \\ I_z \\ I_u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.26}$$

Clearly, the joint invariants are $t,\ u$. Therefore, the invariant transformations are:

$$w = t, Z(w) = u \tag{2.27}$$

Thus, using (2.27), equation (2.3) can be reduced to the ODE:

$$Z'' = 0 \tag{2.28}$$

which has the solution

$$Z(w) = c_1 + c_2\, w \tag{2.29}$$

Thus, the invariant solution of (2.3) is

$$u(t, x, y, z) = c_1 + c_2\, t \tag{2.30}$$

2.7.2.2 Case $\mathcal{L}_2 = \langle V_4, V_5, V_6 \rangle$

Since the reduced row echelon form of the operators of $\mathcal{L}_2$ coincides with that in (2.26), it follows that they have the same solution.

2.8 Invariants solutions in spherical coordinates

In this section, we summarize all the obtained invariant solutions of rank one for the wave equation on Einstein spacetime (2.2) as follows:

I) Solution (2.25):

$$\begin{aligned} u\left(t, r, \theta, \varphi\right) = {} & \mathrm{e}^{\alpha_3\, t} \mathrm{e}^{i\left((\gamma-1)\,\arctan\left(\frac{\cos\theta}{\cos\varphi\sin\theta}\right) - (\nu+\mu-\gamma)\,\arctan\left(\frac{cr\sin\varphi\sin\theta}{\sqrt{1-c^2r^2}}\right)\right)} \\ & Z\left(r^2\left(\cos^2\varphi\sin^2\theta + \cos^2\theta\right)\right) \end{aligned} \tag{2.31}$$

where

$$Z(w) = (c^2w-1)^{\frac{1}{2}\,(\nu+\mu-\gamma)}\,(c^2w)^{\frac{1}{2}\,(\gamma-1)}$$
$$\Big(c_1F(\nu,\mu;\gamma;c^2w) + c_2(c^2w)^{1-\gamma}F(\nu-\gamma+1,\mu-\gamma+1;2-\gamma;c^2w)\Big), \tag{2.32}$$

$F(\nu,\mu;\gamma;r)$ is the hypergeometric function and $\alpha_3^2 = c^2 - c^2(\mu-\nu)^2$.

II) Solution (2.30):

$$u\,(t) = c_1 + c_2\;t \tag{2.33}$$

CHAPTER 3

WAVE EQUATION ON ANTI-EINSTEIN SPACETIME

In this chapter, we will discuss the wave equation $\Delta u = 0$ on the spherically symmetric space time with the metric

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{\mu(t,r)}d\theta^2 - e^{\mu(t,r)}\sin^2\theta d\varphi^2 \tag{3.1}$$

where $\nu = 0,\ \lambda = -\ln(\alpha r^2 + 1)$ and $\mu = \ln r^2,\ \alpha = c^2 > 0$. Hence the corresponding wave equation on this metric is

$$\begin{aligned}&\tfrac{\partial}{\partial t}\left(\tfrac{r^2 u_t \sin\theta}{\sqrt{c^2r^2+1}}\right) - \tfrac{\partial}{\partial r}(r^2 u_r \sin\theta\sqrt{c^2r^2+1}) - \tfrac{\partial}{\partial \theta}\left(\tfrac{u_\theta \sin\theta}{\sqrt{c^2r^2+1}}\right) - \\ &\tfrac{\partial}{\partial \varphi}\left(\tfrac{u_\varphi}{\sin\theta\sqrt{c^2r^2+1}}\right) = 0.\end{aligned} \tag{3.2}$$

Equation (3.2) can be written in Cartesian coordinates as:

$$
\begin{aligned}
&(c^2x^2+1)\,u_{xx}+(c^2y^2+1)\,u_{yy}+(c^2z^2+1)\,u_{zz}+2\,c^2yzu_{yz}\\
&+3\,xc^2u_x+2\,c^2xyu_{xy}+2\,c^2xzu_{xz}+3\,c^2zu_z+3\,c^2yu_y-u_{tt}=0
\end{aligned}
\tag{3.3}
$$

3.1 Lie point symmetries of wave equation on Anti-Einstein spacetime

By using Theorem 1.31 and the isometries of the metric (2.1) given in [21] the Lie symmetry algebra of the wave equation (3.2) consists of the eight-dimensional subalgebra spanned by

$$
\begin{aligned}
X_1 &= [0, A\sin\varphi\sin\theta, \tfrac{A\sin\varphi\cos\theta}{r}, \tfrac{A\cos\varphi}{r\sin\theta}, 0]\\
X_2 &= [0, A\cos\varphi\sin\theta, \tfrac{A\cos\varphi\cos\theta}{r}, -\tfrac{A\sin\varphi}{r\sin\theta}, 0]\\
X_3 &= [0, A\cos\theta, -\tfrac{A\sin\theta}{r}, 0, 0]\\
X_4 &= [0, 0, -\cos\varphi, \tfrac{\sin\varphi}{\tan\theta}, 0]\\
X_5 &= [0, 0, \sin\varphi, \tfrac{\cos\varphi}{\tan\theta}, 0]\\
X_6 &= [0, 0, 0, 1, 0]\\
X_7 &= [1, 0, 0, 0, 0]\\
X_8 &= [0, 0, 0, 0, u]
\end{aligned}
\tag{3.4}
$$

and the infinite-dimensional ideal consisting of the operators

$$
X_\tau = \tau(t, r, \theta, \varphi)\frac{\partial}{\partial u} \tag{3.5}
$$

where $\tau(t, r, \theta, \varphi)$ is an arbitrary solution of the wave equation (3.2) and $A = \sqrt{c^2r^2 + 1}$.

Moreover, the generators (3.4) can be written in Cartesian coordinates as follows:

$$
\begin{aligned}
X_1 &= [0, 0, B, 0, 0] \\
X_2 &= [0, B, 0, 0, 0] \\
X_3 &= [0, 0, 0, B, 0] \\
X_4 &= [0, -z, 0, x, 0] \\
X_5 &= [0, 0, z, -y, 0] \\
X_6 &= [0, -y, x, 0, 0] \\
X_7 &= [1, 0, 0, 0, 0] \\
X_8 &= [0, 0, 0, 0, u].
\end{aligned}
\tag{3.6}
$$

where $B = \sqrt{1 + c^2(x^2 + y^2 + z^2)}$.

3.2 Lie point symmetry transformations

The one-parameter groups $G_i(\varepsilon) = \{e^{\varepsilon X_i} : \varepsilon \in \mathbb{R}\}$ generated by (3.4) are given by solving the initial value problems (1.5) as follows:

$$
\begin{aligned}
&G_1(\varepsilon_1) : (t, x, y, z, u) \mapsto (t, x, \frac{e^{-c\varepsilon_1}\left(2\,cBe^{2\,c\varepsilon_1}y + (B^2 + c^2y^2)e^{2\,c\varepsilon_1} - 1 - c^2\left(x^2 + z^2\right)\right)}{2\,c^2y + 2\,cB}, z, u) \\
&G_2(\varepsilon_2) : (t, x, y, z, u) \mapsto (t, \frac{\left(2\,\mathrm{e}^{2\,c\varepsilon_2}(B^2 - 1) + 2\,cB\mathrm{e}^{2\,c\varepsilon_2}x - c^2(y^2 + z^2) + \mathrm{e}^{2\,c\varepsilon_2} - 1\right)\mathrm{e}^{-c\varepsilon_2}}{2\,c^2x + 2\,cB}, y, z, u) \\
&G_3(\varepsilon_3) : (t, x, y, z, u) \mapsto (t, x, y, \frac{\mathrm{e}^{-c\varepsilon_3}\left(2\,cB\mathrm{e}^{2\,c\varepsilon_3}z + B^2\mathrm{e}^{2\,c\varepsilon_3} - 1 - c^2\left(x^2 + y^2\right)\right)}{2\,c^2z + 2\,cB}, u) \\
&G_4(\varepsilon_4) : (t, x, y, z, u) \mapsto (t, x \sin\varepsilon_4 - z\cos\varepsilon_4, y, x\sin\varepsilon_4 + z\cos\varepsilon_4, u) \\
&G_5(\varepsilon_5) : (t, x, y, z, u) \mapsto (t, x, -z\sin\varepsilon_5 - y\cos\varepsilon_5, y\sin\varepsilon_5 - z\cos\varepsilon_5, u) \\
&G_6(\varepsilon_6) : (t, x, y, z, u) \mapsto (t, -y\sin\varepsilon_6 + x\cos\varepsilon_5, x\sin\varepsilon_6 + y\cos\varepsilon_6, z, u) \\
&G_7(\varepsilon_7) : (t, x, y, z, u) \mapsto (t + \varepsilon_7, x, y, z, u) \\
&G_8(\varepsilon_8) : (t, x, y, z, u) \mapsto (t, x, y, z, u + \varepsilon_8)
\end{aligned}
\tag{3.7}
$$

3.3 Lie Algebra Structure

The non-zero Lie brackets of (3.4) are:

$$
\begin{array}{llll}
[X_1, X_2] = -c^2X_6, & [X_1, X_3] = -c^2X_5, & [X_1, X_5] = -X_3, & [X_1, X_6] = -X_2 \\
[X_2, X_3] = c^2X_4, & [X_2, X_4] = X_3, & [X_2, X_6] = X_1, & [X_3, X_4] = -X_2, \\
[X_3, X_5] = X_1, & [X_4, X_5] = X_6, & [X_4, X_6] = -X_5, & [X_5, X_6] = X_4.
\end{array}
\tag{3.8}
$$

The Levi-Decomposition of this algebra is $\mathcal{L} = \{X_1, X_2, X_3, X_4, X_5, X_6\} \oplus \{X_7, X_8\}$. Let S be the semisimple part of this decomposition.

To determine the structure of the semisimple part we need to find a Cartan algebra and the root space decomposition with respect to the Cartan algebra. In this case, the Killing form is not negative definite and it has exactly three negative eigenvalues. This means that the maximal compact algebra is three-dimensional.

The reason is that if K is maximal compact subalgebra of the Lie algebra $\mathcal{L}$, then any compact subagebra of $\mathcal{L}$ is conjugate to a subalgebra of K. Moreover, every one-dimensional subalgebra of K is conjugate to a subalgebra of a fixed Cartan subalgebra of K [12], [11] [17].

As we will explain later, the subalgebra $\langle X_4, X_5, X_6\rangle$ is a copy of $so(3)$ in the given Lie algebra. Thus $K = \langle X_4, X_5, X_6\rangle$ is a maximal compact subalgebra of the algebra S. A Cartan subalgebra of S can be obtained by choosing any element of K and computing its centralizer. We choose, for example, X_6 as a representative of a Cartan algebra of K. Computing the centralizer of X_6, we find that it is $\langle X_3, X_6\rangle$. Also, as the centralizer of $\langle X_3, X_6\rangle$ is itself, $C = \langle X_3, X_6\rangle$ is a Cartan subalgebra of S. Moreover, computing the eigenvalues of X_3, we find that all eigenvalues of $\mathrm{ad}X_3$ are real and X_3 is diagonalizable. Moreover, the centralizer of X_3 is C and the centralizer of X_6 is also C , this means that C is the maximally real Cartan subalgebra.

We find roots of C in S. The roots are $(c, i), (c, -i), (-c, i), (-c, -i)$, the positive roots are $(c, i), (c, -i)$ and clearly the sum of these positive roots is not a root. The root spaces for the positive roots (c, i) and $(c, -i)$ are $\langle X_1 + cX_5 + i(cX_4 - X_2)\rangle$. Let $N = \langle X_1 + cX_5, X_2 - cX_4\rangle$. The algebra $B = A \oplus N$, where

$A = \langle X_3 \rangle$ is the real part of C, has the property that every solvable algebra with real eigenvalues in the adjoint representation is conjugate to a subalgebra of B - as mentioned in section 1.3.1.3. We compute the normalizers of each conjugacy class of N. The normalizer of each representative element of the one-dimensional optimal system of N does not contain a Cartan algebra. Therefore, we keep only N because its normalizer $\mathcal{N}(N)$ is solvable and contains a Cartan algebra. Thus, there is only one maximal solvable subalgebra, namely $\mathcal{N}(N) = \langle X_6, X_3 \rangle \oplus \langle X_1 + cX_5, X_2 - cX_4 \rangle$. Therfore, the Iwasawa decomposition of S is $K \oplus A \oplus N = \langle X_4, X_5, X_6 \rangle \oplus \langle X_3 \rangle \oplus \langle X_1 + cX_5, X_2 - cX_4 \rangle$ [12, 17].

This gives us the following change of basis which makes the computations easier:

$$
\begin{aligned}
&V_1 = X_4, && V_2 = X_5, && V_3 = X_6, && V_4 = X_3, \\
&V_5 = X_1 + cX_5, && V_6 = X_2 - cX_4, && V_7 = X_7, && V_8 = X_8.
\end{aligned}
\tag{3.9}
$$

The Lie brackets of (3.9) are

$$
\begin{aligned}
&[V_1, V_2] = V_3, && [V_1, V_3] = -V_2, && [V_1, V_4] = cV_1 + V_6, && [V_1, V_5] = cV_3, \\
&[V_1, V_6] = -V_4, && [V_2, V_3] = V_1, && [V_2, V_4] = cV_2 - V_5, && [V_2, V_5] = V_4, \\
&[V_2, V_6] = cV_3 && [V_3, V_5] = V_6, && [V_3, V_6] = -V_5, && [V_4, V_5] = cV_5, \\
&[V_4, V_6] = cV_6.
\end{aligned}
\tag{3.10}
$$

In fact, the semisimple part S is isomorphic to $so(1,3)$ as can be seen by working with its Cartan algebra and the associated root space decompositions. The algebra $\langle V_7, V_8 \rangle$ is the center of the Lie algebra $\mathcal{L}$.

3.4 Optimal systems of solvable subalgebras of the semisimple part $so(1,3)$

To find the optimal system of $so(1,3)$, we first find the one-dimensional optimal system and a rough classification of higher order subalgebras inside the maximal solvable subalgebra spanned by $E_1 := V_3, E_2 := V_4, E_3 := V_5, E_4 := V_6$. The corresponding non-zero Lie brackets of this subalgebra are:

$$[E_1, E_3] = E_4, \quad [E_1, E_4] = -E_3, \quad [E_2, E_3] = cE_3, \quad [E_2, E_4] = cE_4. \tag{3.11}$$

Secondly, we obtain the higher-dimensional optimal system by removing the repetitions from the obtained rough classification of subalgebras using the adjoint action of $so(1,3)$.

3.4.1 One-dimensional optimal system of the maximal solvable subalgebra of $so(1,3)$

The adjoint action of the Lie algebra with the non-zero Lie brackets (3.11) is represented as

$$A(\varepsilon_1, \ldots, \varepsilon_4) = A_1(\varepsilon_1) \ldots A_4(\varepsilon_4), \tag{3.12}$$

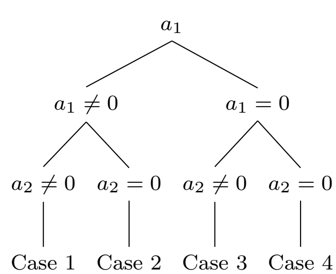
Therefore,

$$A(\varepsilon_1,\ldots,\varepsilon_4) = \begin{pmatrix} 1 & 0 & -\varepsilon_4 & \varepsilon_3 \\ 0 & 1 & c\varepsilon_3 & c\varepsilon_4 \\ 0 & 0 & \mathrm{e}^{-c\varepsilon_2}\cos\varepsilon_1 & -\mathrm{e}^{-c\varepsilon_2}\sin\varepsilon_1 \\ 0 & 0 & \mathrm{e}^{-c\varepsilon_2}\sin\varepsilon_1 & \mathrm{e}^{-c\varepsilon_2}\cos\varepsilon_1 \end{pmatrix}$$

Now, the system of invariants $\phi(a_1, a_2, a_3, a_4)$ is given by:

$$a_4\phi_{a_3} - a_3\phi_{a_4} = 0, \quad a_3\phi_{a_3} + a_4\phi_{a_4} = 0, \quad a_1\phi_{a_4} + a_2c\phi_{a_3} = 0, \quad a_1\phi_{a_3} - a_2c\phi_{a_4} = 0 \tag{3.13}$$

By solving the system (3.13), we get the invariants $\{a_1, a_2\}$. For each invariant, we will discuss two cases: either it is zero or not. In case it is zero, solving system (3.13) will give extra invariants. By repeating this process, we get four cases, as the following tree indicates:



The values of $\varepsilon_j, j = 1, \ldots, 4$ and the representative element for each case is the following:

- Case 1: $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a_2\,ca_3 + a_1\,a_4}{a_2{}^2c^2 + a_1{}^2}, \varepsilon_4 = \frac{a_1\,a_3 - a_2\,ca_4}{a_2{}^2c^2 + a_1{}^2}\right\}$, the representative element is $E_1 + \alpha E_2,\ \alpha \neq 0$.

- Case 2: $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a_4}{a_1}, \varepsilon_4 = \frac{a_3}{a_1}\right\}$, the representative element is E_1.

- Case 3: $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a_3}{a_2\, c}, \varepsilon_4 = -\frac{a_4}{a_2\, c}\right\}$, the representative element is E_2.

- Case 4: $\left\{\varepsilon_1 = \arcsin\left(\frac{a_4}{\sqrt{{a_3}^2+{a_4}^2}}\right), \varepsilon_2 = \frac{\ln\left({a_3}^2+{a_4}^2\right)}{2c}\right\}$, the representative element is E_3.

3.4.2 Two-dimensional rough classification of the maximal solvable subalgebra of $so(1,3)$

After constructing the one-dimensional optimal system, we will use it to get the two-dimensional rough classification by using the expansion method.

1. $\mathcal{N}(A)/\langle A\rangle = \langle \bar{E}_2\rangle$ for $A = E_1 + \alpha E_2$:

   The one-dimensional optimal system in $\mathcal{N}(A)/\langle A\rangle$ is E_2.

2. $\mathcal{N}(A)/\langle A\rangle = \langle \bar{E}_2\rangle$ for $A = E_1$:

   The one-dimensional optimal system in $\mathcal{N}(A)/\langle A\rangle$ is E_2.

3. $\mathcal{N}(A)/\langle A\rangle = \langle \bar{E}_1\rangle$ for $A = E_2$:

   The one-dimensional optimal system in $\mathcal{N}(A)/\langle A\rangle$ is E_1.

4. $\mathcal{N}(A)/\langle A\rangle = \langle \bar{E}_2, \bar{E}_4\rangle$ for $A = E_3$:

   To find a one-dimensional optimal system of $\mathcal{N}(A)/\langle A\rangle$, choose a general element as

$$X = a_1\bar{E}_2 + a_2\bar{E}_4.$$

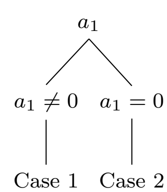
The adjoint matrix in this case is :

$$A(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} 1 & c\varepsilon_2\, \mathrm{e}^{-c\varepsilon_1} \\ & \\ 0 & \mathrm{e}^{-c\varepsilon_1} \end{pmatrix}$$

and the invariant system is:

$$\phi_{a_2} = 0. \tag{3.14}$$

By repeating the preceding procedure, the values of $\varepsilon_1, \varepsilon_2$ and the representative element for each case can be given in the following tree:



- Case 1: $\left\{\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = -\frac{a_2}{a_1\, c}, \varepsilon_4 = 0\right\}$, the result is $\{E_2\}$.
- Case 2: $\left\{\varepsilon_1 = \frac{\ln(a_2)}{c}, \varepsilon_2 = 0, \varepsilon 3 = 0, \varepsilon_4 = 0\right\}$, the result is $\{E_4\}$.

It is clear that applying the expansion method for all the representative elements of one-dimensional optimal system of $so(1,3)$ leads to the two-dimensional subalgebra $\{\langle E_1, E_2\rangle, \langle E_2, E_3\rangle, \langle E_3, E_4\rangle\}$.

3.4.3 Three-dimensional rough classification of the maximal solvable subalgebra of $so(1,3)$

Similarly, applying the expansion method for two-dimensional rough classification of the maximal solvable subalgebra of $so(1,3)$ implies that the $\mathcal{N}(A)/\langle A\rangle$ is non-empty only for $A = \langle E_3, E_4\rangle$. Since $\mathcal{N}(\langle E_3, E_4\rangle)/\langle E_3, E_4\rangle = \langle E_1, E_2\rangle$ is abelian, then the one-dimensional optimal system of $\langle E_1, E_2\rangle$ is: $E_1 + \alpha E_2, \alpha \neq 0,\ E_1,\ E_2$. Therefore, the three-dimensional rough classification of the maximal solvable subalgebra of $so(1,3)$ is $\{\langle E_2, E_3, E_4\rangle\} \cup \{\langle E_1 + \alpha E_2, E_3, E_4\rangle : \alpha \in \mathbb{R}\}$.

Finally, in order to get the optimal systems of solvable subalgebras of $so(1,3)$, one can remove the repetitions in the obtained one-dimensional optimal system and the higher-dimensional rough classification of the maximal solvable using the adjoint action of $so(1,3)$. However, for subalgebras with dimension greater than one, we have taken a simpler approach to remove the repetitions by computing the invariant tools like normalizers, centralizers and the eigenvalues of X/X'.

Theorem 3.1 *The optimal systems Θ_i up to order three of the solvable subalgebras of $so(1,3)$ with the non-zero Lie brackets (3.10) are the following:*

- *The 1-dimensional solvable optimal system Θ_1 is*

  $\{\langle V_4\rangle,\ \langle V_5\rangle\} \cup \{\langle V_3 + \alpha V_4\rangle : \alpha \in \mathbb{R}\}$.

- *The 2-dimensional solvable optimal system Θ_2 is* $\{\langle V_3, V_4\rangle, \langle V_4, V_5\rangle, \langle V_5, V_6\rangle\}$.

- *The 3-dimensional solvable optimal system Θ_3 is*

  $\{\langle V_4, V_5, V_6\rangle\} \cup \{\langle V_3 + \alpha V_4, V_5, V_6\rangle : \alpha \in \mathbb{R}\}$.

Proof. To remove the repetitions in the obtained one-dimensional optimal system and the higher-dimensional rough classification of the maximal solvable subalgebra of $\mathcal{L}$, we use their normalizers in $so(1,3)$ as follows:

- The one-dimensional optimal system of the maximal solvable subalgebra of $so(1,3)$ is itself the one-dimensional optimal system of $so(1,3)$. This is because the representative elements are non-conjugate under the adjoint action of $so(1,3)$, as can be seen using the action of corresponding adjoint group given by

  $$A(\varepsilon_1,\ldots,\varepsilon_6) = A_1(\varepsilon_1)\ldots A_6(\varepsilon_6), \tag{3.15}$$

  where $A_i(\varepsilon_i)$ are defined in Remark 1.10.

- The two-dimensional abelian subalgebras are $\langle V_3, V_4\rangle, \langle V_5, V_6\rangle$. The non-abelian subalgebra $\langle V_4, V_5\rangle$ is clearly non-conjugate with both of them. Moreover, since the normalizers of the two-dimensional abelian subalgebras are $\mathcal{N}(\langle V_3, V_4\rangle)/\langle V_3, V_4\rangle = 0$, $\mathcal{N}(\langle V_5, V_6\rangle)/\langle V_5, V_6\rangle = \langle \bar{V}_3, \bar{V}_4\rangle$. As their dimensions are different, they are non-conjugate.

- All the three-dimensional subalgebras given in the rough classification have the same normalizers, centralizers and commutators, namely the abelian subalgebra $\langle V_5, V_6\rangle$.

  Let X be one of these algebras. We find that the eigenvalues of X/X' are repeated real in one case, purely imaginary in one case and complex conjugates but not purely imaginary in the third case. Therefore, they are non-comjugate.

∎

3.5 Optimal systems of solvable subalgebras of $\mathcal{L} = so(1,3) \oplus \mathbb{R}^2$

Clearly, the 1-dimensional optimal system $\tilde{\Theta}_1$ of $\mathbb{R}^2$ is $\{\langle V_8\rangle\} \cup \{\langle V_7+\alpha V_8\rangle : \alpha \in \mathbb{R}\}$ and the only 2-dimensional optimal system $\tilde{\Theta}_2$ of $\mathbb{R}^2$ is $\{\langle V_7, V_8\rangle\}$.

In order to get the optimal systems of the full Lie algebra up to order three, we use the optimal systems of $so(1,3)$ constructed in Theorem 3.1. We join each one of them with the optimal systems of the abelian algebra $\mathbb{R}^2$ as explained in Section 2.5:

- To get the one-dimensional optimal system of $\mathcal{L}$,
  - i) either we take $\tilde{\Theta}_1$ itself, in this case we get $\{\langle V_8\rangle\} \cup \{\langle V_7+\beta V_8\rangle : \beta \in \mathbb{R}\}$;
  - ii) or we add an arbitrary element from $\mathbb{R}^2$ to every representative element in Θ_1, in this case we get $\{\langle V_4 + Z_1\rangle,\ \langle V_5 + Z_1\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1\rangle : \alpha \in \mathbb{R}\}$.
- To get the two-dimensional optimal system of $\mathcal{L}$,
  - i) either we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_2, in this case we get $\{\langle V_3+Z_1, V_4+Z_2\rangle, \langle V_4+Z_1, V_5+Z_2\rangle, \langle V_5+Z_1, V_6+Z_2\rangle\}$,

ii) or we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_1 and combine the result with an element from $\tilde{\Theta}_1$, in this case we get $\{\langle V_4 + Z_1, Z_3\rangle, \langle V_5 + Z_1, Z_3\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1, Z_3\rangle : \alpha \in \mathbb{R}\}$.

iii) or take $\tilde{\Theta}_2$ itself, in this case we get $\{\langle V_7, V_8\rangle\}$.

- To get the three-dimensional optimal system of $\mathcal{L}$,

i) either we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_3, in this case we get $\{\langle V_4 + Z_1, V_5 + Z_2, V_6 + Z_3\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1, V_5 + Z_2, V_6 + Z_3\rangle : \alpha \in \mathbb{R}\}$,

ii) or we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_2 and combine the result with an element from $\tilde{\Theta}_1$, in this case we get $\{\langle V_3 + Z_1, V_4 + Z_2, Z_3\rangle, \langle V_4 + Z_1, V_5 + Z_2, Z_3\rangle, \langle V_5 + Z_1, V_6 + Z_2, Z_3\rangle\}$,

iii) or we add an arbitrary element from $\mathbb{R}^2$ to every element in Θ_1 and combine the result with an element from $\tilde{\Theta}_2$, in this case we get $\{\langle V_4 + Z_1, V_7, V_8\rangle, \langle V_5 + Z_1, V_7, V_8\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1, V_7, V_8\rangle : \alpha \in \mathbb{R}\}$,

where $Z_1 = \alpha_1 V_7 + \beta_1 V_8, Z_2 = \alpha_2 V_7 + \beta_2 V_8$ are arbitrary elements of $\mathbb{R}^2$ and $Z_3 = V_7 + \alpha_3 V_8$ or $Z_3 = V_8$ represents a one-dimensional optimal system of $\mathbb{R}^2$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mathbb{R}$.

Finally, we check that the obtained class is a subalgebra by taking the wedge product of its commutator with each element in the class and make these wedges equal to zero. Therefore, we have the following theorem

Theorem 3.2 *The optimal systems of solvable subalgebras of* $\mathcal{L}$ *with the non-zero Lie brackets (3.10) are as follows:*

- *The* 1*-dimensional solvable optimal system is* $\{\langle V_4 + Z_1\rangle,\ \langle V_5 + Z_1\rangle, \langle V_8\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1\rangle, \langle V_7 + \beta V_8\rangle : \alpha, \beta \in \mathbb{R}\}$.

- *The* 2*-dimensional solvable optimal system is* $\{\langle V_3 + Z_1, V_4 + Z_2\rangle, \langle V_4 + Z_1, V_5\rangle, \langle V_5 + Z_1, V_6 + Z_2\rangle, \langle V_4 + Z_1, Z_3\rangle, \langle V_5 + Z_1, Z_3\rangle, \langle V_7, V_8\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1, Z_3\rangle : \alpha \in \mathbb{R}\}$.

- *The* 3*-dimensional solvable optimal system is* $\{\langle V_4 + Z_1, V_5, V_6\rangle, \langle V_3 + Z_1, V_4 + Z_2, Z_3\rangle, \langle V_4 + Z_1, V_5, Z_3\rangle, \langle V_5 + Z_1, V_6 + Z_2, Z_3\rangle, \langle V_4 + Z_1, V_7, V_8\rangle, \langle V_5 + Z_1, V_7, V_8\rangle\} \cup \{\langle V_3 + \alpha V_4 + Z_1, V_5, V_6\rangle, \langle V_3 + \alpha V_4 + Z_1, V_7, V_8\rangle : \alpha \in \mathbb{R}\}$.

Here, $Z_1 = \alpha_1 V_7 + \beta_1 V_8, Z_2 = \alpha_2 V_7 + \beta_2 V_8$ *are arbitrary elements of* $\mathbb{R}^2$ *and* $Z_3 = V_7 + \alpha_3 V_8$ *or* $Z_3 = V_8$ *represents a one-dimensional optimal system of* $\mathbb{R}^2$.

3.6 Optimal systems of non-solvable subalgebras of $\mathcal{L} = so(1,3) \oplus \mathbb{R}^2$

If H is a three-dimensional non-solvable algebra, then H equals its commutator. As the commutator of $\mathcal{L}$ is $so(1,3)$, all such subalgebras of $\mathcal{L}$ are subalgebras of $so(1,3)$. To find such subalgebras we follow the algorithm given in section 1.3.1.3.

We need to construct the copies of $so(3)$ and $sl(2,\mathbb{R})$, if any, by following the algorithm that is given in section 1.3.1.3:

- First, construct the copies of $so(3)$: The element V_3 has the eigenvector $V_1 + iV_2$ corresponding to the eigenvalue $-i$. Therefore, $\tilde{V_3} = -V_3$ has the same eigenvector with the eigenvalue i. Moreover, $[V_1, V_2] = -\tilde{V_3}$. Therefore, $\langle V_1, V_2, V_3\rangle$ forms a copy of $so(3)$.

- The only non-abelian two-dimensional subalgebra in $so(1,3)$ is $\langle V_4, V_5\rangle$ with $[V_4, V_5] = cV_5$. Moreover, the eigenvector of $\mathrm{ad}V_4$ corresponding to the eigenvalue $-c$ is $V_5 - 2cV_2$ and $[V_5 - 2cV_2, V_5] = -2cV_4$. Hence, the subalgebra $\langle V_4, V_5, V_5 - 2cV_2\rangle$ forms a copy of $sl(2, \mathbb{R})$.

This proves the theorem

Theorem 3.3 *The* 3*-dimensional non-solvable optimal system is* $\{\langle V_1, V_2, V_3\rangle, \langle V_4, V_5, V_5 - 2cV_2\rangle\}$*, where the* V_i $(i = 1, \ldots, 6)$ *form a basis of* $so(1,3)$ *given in (3.9).*

3.7 Joint invariants and invariant solutions of three-dimensional optimal systems of $\mathcal{L} = so(1,3) \oplus \mathbb{R}^2$

3.7.1 Solvable subalgebras of $\mathcal{L}$

3.7.1.1 Case $\mathcal{L}_1 = \langle V_3 + \alpha V_4 + Z_1, V_5, V_6 \rangle, \alpha \neq 0$

The generators of $\mathcal{L}_1$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_3 + \alpha V_4 + Z_1 &= [\alpha_1, -y, x, \alpha\sqrt{1 + c^2(x^2+y^2+z^2)}, \beta_1 u], \\ V_5 &= [0, 0, \sqrt{1 + c^2(x^2+y^2+z^2)} + cz, -cy, 0], \\ V_6 &= [0, \sqrt{1 + c^2(x^2+y^2+z^2)} + cz, 0, -cx, 0] \end{aligned} \tag{3.16}$$

The transversality condition (1.43) of (3.16) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_1$ is $\langle V_5, V_6 \rangle$ which is abelian, one can find the invariants of $\mathcal{L}_1$ by starting with V_5 or V_6. The invariants of V_5 are:

$$m_1 = t, m_2 = x, m_3 = \frac{-cz + \sqrt{1 + c^2(x^2+y^2+z^2)}}{1 + c^2(x^2+y^2)}, m_4 = u \tag{3.17}$$

The operator V_6 can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$V_6 = [0, \tfrac{1}{m_3}, 0, 0] \tag{3.18}$$

Next, the invariants of V_6 are

$$n_1 = m_1, n_2 = m_3, n_3 = m_4 \tag{3.19}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_3 + \alpha V_4 + Z_1 = [\alpha_1, -\alpha\, c n_2, \beta_1 n_3] \tag{3.20}$$

We have to study the following two cases:

- Case 1: If $\alpha_1 \neq 0$, the invariants of $V_3 + \alpha V_4 + Z_1$ are

$$n_2\ \mathrm{e}^{\frac{c\alpha}{\alpha_1} n_1}, n_3\ \mathrm{e}^{-\frac{\beta_1}{\alpha_1} n_1} \tag{3.21}$$

Writing the invariants (3.21) in terms of the original variables gives the joint invariants of $\mathcal{L}_1$ as

$$\frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)} \mathrm{e}^{\frac{\alpha\, c}{\alpha_1} t}, u\mathrm{e}^{-\frac{\beta_1}{\alpha_1} t}$$

Therefore, the invariant transformations are:

$$w = \frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)} \mathrm{e}^{\frac{\alpha\, c}{\alpha_1} t},\ \ Z(w) = u\ e^{-\frac{\beta_1}{\alpha_1} t} \tag{3.22}$$

Thus, using (3.22), equation (3.3) can be reduced to the ODE:

$$c\left(\left(\alpha^2+{\alpha_1}^2\right)c+2\,\alpha\,\beta_1\right)w\;Z'+c^2\left(\alpha^2-{\alpha_1}^2\right)w^2\;Z''+{\beta_1}^2\;Z=0 \quad (3.23)$$

which has the non-trivial solution for the following cases:

i) $\alpha^2-\alpha_1^2\neq 0$:

$$Z(w)=c_1\,w^{\frac{-c{\alpha_1}^2-\alpha\,\beta_1+\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{\left(\alpha^2-{\alpha_1}^2\right)c}}+c_2\,w^{\frac{-c{\alpha_1}^2-\alpha\,\beta_1-\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{\left(\alpha^2-{\alpha_1}^2\right)c}} \quad (3.24)$$

ii) $\alpha^2-\alpha_1^2=0,\ c\alpha_1+\beta_1\neq 0$:

$$Z(w)=c_1\,w^{-\frac{\beta_1^2}{2c\alpha_1\,(\alpha_1\,c+\beta_1)}} \quad (3.25)$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z)=Z\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\mathrm{e}^{\frac{\alpha\,c}{\alpha_1}t}\right)e^{\frac{\beta_1}{\alpha_1}t} \quad (3.26)$$

- Case 2: If $\alpha_1=0$, the invariants of $V_3+\alpha V_4+Z_1$ are

$$n_1,\, n_3 n_2^{\frac{\beta_1}{\alpha\,c}} \quad (3.27)$$

Writing the invariants (3.27) in terms of the original variables gives the joint

invariants of $\mathcal{L}_1$ as

$$t, u\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\right)^{\frac{\beta_1}{\alpha\, c}}$$

Therefore, the invariant transformations are:

$$w=t,\ \ Z(w)=u\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\right)^{\frac{\beta_1}{\alpha\, c}} \tag{3.28}$$

Thus, using (3.28), equation (3.3) can be reduced to the ODE:

$$\alpha^2 Z''-\ \beta_1\ \left(\beta_1+2\alpha\, c\right) Z=0 \tag{3.29}$$

which has the solution

$$Z\left(w\right)=\ c_1\, \mathrm{e}^{\frac{\sqrt{\beta_1^2+2\beta_1\,\alpha\, c}}{\alpha}w}+c_2\, \mathrm{e}^{-\frac{\sqrt{\beta_1^2+2\beta_1\,\alpha\, c}}{\alpha}w} \tag{3.30}$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z)=Z(t)\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\right)^{-\frac{\beta_1}{\alpha\, c}} \tag{3.31}$$

3.7.1.2 Case $\mathcal{L}_2 = \langle V_3 + Z_1, V_5, V_6 \rangle$

The generators of $\mathcal{L}_2$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_3 + Z_1 &= [\alpha_1, -y, x, 0, \beta_1 u], \\ V_5 &= [0, 0, \sqrt{1 + c^2(x^2 + y^2 + z^2)} + cz, -cy, 0], \\ V_6 &= [0, \sqrt{1 + c^2(x^2 + y^2 + z^2)} + cz, 0, -cx, 0] \end{aligned} \tag{3.32}$$

The transversality condition (1.43) of (3.32) with rank three is satisfied for $\alpha_1 \neq 0$. Since the derived Lie algebra generated by $\mathcal{L}_2$ is $\langle V_5, V_6 \rangle$ which is abelian, one can find the invariants of $\mathcal{L}_2$ by starting with the invariants of $\langle V_5, V_6 \rangle$ which are given by (3.19) as

$$n_1 = t, n_2 = \frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\left(x^2 + y^2\right)}, n_3 = u \tag{3.33}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_3 + Z_1 = [\alpha_1, 0, \beta_1 n_3] \tag{3.34}$$

Finally, the invariants of $V_3 + Z_1$ are

$$n_2, n_3 \ \mathrm{e}^{-\frac{\beta_1}{\alpha_1} n_1} \tag{3.35}$$

Writing the invariants (3.35) in terms of the original variables gives the joint invariants of $\mathcal{L}_2$ as

$$\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\,(x^2+y^2)},\, u\mathrm{e}^{-\frac{\beta_1}{\alpha_1}t}$$

Therefore, the invariant transformations are:

$$w=\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)},\;\; Z(w)=u\mathrm{e}^{-\frac{\beta_1}{\alpha_1}t} \tag{3.36}$$

Thus, using (3.36), equation (3.3) can be reduced to the Cauchy Euler ODE:

$$w^2Z''-wZ'-\Big(\frac{\beta_1}{c\alpha_1}\Big)^2Z=0 \tag{3.37}$$

which has the solution

$$Z\,(w)=c_1\,w^{\frac{c\alpha_1+\sqrt{c^2{\alpha_1}^2+{\beta_1}^2}}{c\alpha_1}}+c_2\,w^{\frac{c\alpha_1-\sqrt{c^2{\alpha_1}^2+{\beta_1}^2}}{c\alpha_1}} \tag{3.38}$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z)=Z\Big(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}\Big)\mathrm{e}^{\frac{\beta_1}{\alpha_1}t} \tag{3.39}$$

3.7.1.3 Case $\mathcal{L}_3 = \langle V_4 + Z_1, V_5, V_6 \rangle$

The generators of $\mathcal{L}_3$ in Cartesian coordinates are as follows:

$$
\begin{aligned}
V_4 + Z_1 &= [\alpha_1, 0, 0, \sqrt{1 + c^2(x^2 + y^2 + z^2)}, \beta_1 u], \\
V_5 &= [0, 0, \sqrt{1 + c^2(x^2 + y^2 + z^2)} + cz, -cy, 0], \\
V_6 &= [0, \sqrt{1 + c^2(x^2 + y^2 + z^2)} + cz, 0, -c\,x, 0]
\end{aligned}
\tag{3.40}
$$

The transversality condition (1.43) of (3.40) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_3$ is $\langle V_5, V_6 \rangle$ which is abelian, one can find the invariants of $\mathcal{L}_3$ by starting with the invariants of $\langle V_5, V_6 \rangle$ which are given by (3.19) as

$$
n_1 = t, n_2 = \frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)}, n_3 = u \tag{3.41}
$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$
V_4 + Z_1 = [\alpha_1, -c\; n_2, \beta_1 n_3] \tag{3.42}
$$

We have to consider the following two cases:

- <u>Case 1</u>: If $\alpha_1 \neq 0$, the invariants of $V_4 + Z_1$ are

$$
n_2 \mathrm{e}^{\frac{c}{\alpha_1} n_1}, n_3 \mathrm{e}^{-\frac{\beta_1}{\alpha_1} n_1} \tag{3.43}
$$

Writing the invariants (3.43) in terms of the original variables gives the joint

invariants of $\mathcal{L}_3$ as

$$\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\mathrm{e}^{\frac{ct}{\alpha_1}}, u\mathrm{e}^{-\frac{\beta_1}{\alpha_1}t} \tag{3.44}$$

Therefore, the invariant transformations are:

$$w=\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\mathrm{e}^{\frac{ct}{\alpha_1}},\ Z(w)=u\ e^{-\frac{\beta_1}{\alpha_1}t} \tag{3.45}$$

Thus, using (3.45), equation (3.3) can be reduced to the ODE:

$$c^2\left(\alpha_1^2-1\right)w^2Z''-\left(\left(1+{\alpha_1}^2\right)c+2\,\beta_1\right)cwZ'-{\beta_1}^2Z=0 \tag{3.46}$$

which has the non-trivial solution for the following cases:

i) $\alpha_1^2-1\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{c{\alpha_1}^2+\beta_1+\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\beta_1+{\beta_1}^2}}{c\left({\alpha_1}^2-1\right)}}+c_2\,w^{\frac{c{\alpha_1}^2+\beta_1-\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\beta_1+{\beta_1}^2}}{c\left({\alpha_1}^2-1\right)}} \tag{3.47}$$

ii) $\alpha_1^2-1=0,\ c+\beta_1\neq 0$:

$$Z\left(w\right)=c_1\,w^{-\frac{\beta_1^2}{2c\left(c+\beta_1\right)}} \tag{3.48}$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z)=Z\Big(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2\left(x^2+y^2\right)}\mathrm{e}^{\frac{ct}{\alpha_1}}\Big)\ e^{\frac{\beta_1}{\alpha_1}t} \tag{3.49}$$

- <u>Case 2</u>: If $\alpha_1 = 0$, the invariants of $V_4 + Z_1$ are

$$n_1, n_3\ n_2{}^{\frac{\beta_1}{c}} \tag{3.50}$$

Writing the invariants (3.50) in terms of the original variables gives the joint invariants of $\mathcal{L}_3$ as

$$t, u\left(\frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)}\right)^{\frac{\beta_1}{c}} \tag{3.51}$$

Therefore, the invariant transformations are:

$$w = t,\ \ Z(w) = u\left(\frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)}\right)^{\frac{\beta_1}{c}} \tag{3.52}$$

Thus, using (3.52), equation (3.3) can be reduced to the Cauchy Euler ODE:

$$Z'' - \left(2\,\beta_1\,c + {\beta_1}^2\right) Z = 0 \tag{3.53}$$

which has the solution

$$Z\,(w) = c_1\,\mathrm{e}^{\sqrt{\beta_1^2 + 2\,c\beta_1}w} + c_2\,\mathrm{e}^{-\sqrt{\beta_1^2 + 2\,c\beta_1}w} \tag{3.54}$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z) = Z(t)\ \left(\frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2\,(x^2 + y^2)}\right)^{-\frac{\beta_1}{c}} \tag{3.55}$$

3.7.1.4 Case $\mathcal{L}_4 = \langle V_3 + Z_1, V_4 + Z_2, Z_3\rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_4$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_3 + Z_1 &= [\alpha_1, -y, x, 0, \beta_1\ u], \\ V_4 + Z_2 &= [\alpha_2, 0, 0, \sqrt{1 + c^2(x^2 + y^2 + z^2)}, \beta_2\ u], \\ Z_3 &= [1, 0, 0, 0, \alpha_3\ u] \end{aligned} \tag{3.56}$$

The transversality condition (1.43) of (3.56) with rank three is always satisfied. Since the Lie algebra generated by $\mathcal{L}_4$ is abelian, one can find the invariants of $\mathcal{L}_4$ in any order. The invariants of Z_3 are:

$$m_1 = x, m_2 = y, m_3 = z, m_4 = u\mathrm{e}^{-\alpha_3 t} \tag{3.57}$$

The operators can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$\begin{aligned} V_3 + Z_1 &= [-m_2, m_1, 0, (\beta_1 - \alpha_3\,\alpha_1)\, m_4], \\ V_4 + Z_2 &= [0, 0, \sqrt{1 + c^2({m_1}^2 + {m_2}^2 + {m_3}^2)}, (\beta_2 - \alpha_2\,\alpha_3)\, m_4] \end{aligned} \tag{3.58}$$

Next, the invariants of $V_3 + Z_1$ are

$$n_1 = {m_1}^2 + {m_2}^2, n_2 = m_3, n_3 = m_4 \mathrm{e}^{(\beta_1 - \alpha_3\,\alpha_1)\arctan\left(\frac{m_1}{m_2}\right)} \tag{3.59}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_4 + Z_2 = [0, \sqrt{1 + c^2(n_1 + {n_2}^2)}, (\beta_2 - \alpha_2\,\alpha_3)\, n_3] \tag{3.60}$$

Finally, the invariants of $V_4 + Z_2$ are

$$n_1, n_3 \left(c\, n_2 + \sqrt{1 + c^2(n_1 + {n_2}^2)}\right)^{\frac{\alpha_2\,\alpha_3 - \beta_2}{c}} \tag{3.61}$$

Writing the invariants (3.61) in terms of the original variables gives the joint invariants of $\mathcal{L}_4$ as

$$x^2 + y^2, u\mathrm{e}^{-\alpha_3\, t}\mathrm{e}^{A_1\,\arctan\left(\frac{x}{y}\right)}\left(cz + \sqrt{1 + c^2\left(x^2 + y^2 + z^2\right)}\right)^{A_2} \tag{3.62}$$

where $A_1 = (\beta_1 - \alpha_3\,\alpha_1)\,, A_2 = \frac{\alpha_2\,\alpha_3 - \beta_2}{c}$

Therefore, the invariant transformations are:

$$w = x^2 + y^2,\ Z(w) = u\mathrm{e}^{-\alpha_3\, t}\mathrm{e}^{A_1\,\arctan\left(\frac{x}{y}\right)}\left(cz + \sqrt{1 + c^2\left(x^2 + y^2 + z^2\right)}\right)^{A_2} \tag{3.63}$$

Thus, using (3.63), equation (3.3) can be reduced to the ODE:

$$4(w^2 + c^2w^3)Z'' + 4(w - c^2\left(A_2 - 2\right)w^2)Z' + \left(\ \left(\left({A_2}^2 - 2\,A_2\right)c^2 - \alpha^2\right)w + {A_1}^2\right)Z = 0 \tag{3.64}$$

which can be transformed using the transformation $w = -\frac{r}{c^2}$,

$Z(w) = r^{\frac{1}{2}\,iA_1}R(r)$ to the hypergeometric differential equation

$$r(r-1)R'' + \big((\nu + \mu + 1)r - \gamma\big)R' + \nu\mu R = 0 \tag{3.65}$$

with $\nu = \frac{(1 + iA_1 - A_2)c + \sqrt{c^2 + {\alpha_3}^2}}{2c}, \mu = \frac{(1 + iA_1 - A_2)c - \sqrt{c^2 + {\alpha_3}^2}}{2c}, \gamma = 1 + iA_1$. The solution of (3.65) is given in terms of the hypergeometric function $F(\nu, \mu; \gamma; r)$ as

$$R(r) = c_1 F(\nu, \mu; \gamma; r) + c_2 r^{1-\gamma} F(\nu - \gamma + 1, \mu - \gamma + 1; 2 - \gamma; r) \tag{3.66}$$

Therefore, the solution of (3.64) is

$$Z(w) = (-c^2 w)^{\frac{1}{2}(\gamma-1)} R(-c^2 w) \tag{3.67}$$

Thus, the invariant solution of (3.3) is

$$u(t,x,y,z) = Z(x^2+y^2)\, \mathrm{e}^{\alpha_3 t} \mathrm{e}^{i(\gamma-1)\arctan\left(\frac{x}{y}\right)} \left(cz + \sqrt{1 + c^2\left(x^2+y^2+z^2\right)}\right)^{3\nu-\gamma-\mu} \tag{3.68}$$

where $\alpha_3^2 = c^2(\nu-\mu)^2 - c^2$.

3.7.1.5 Case $\mathcal{L}_5 = \langle V_4 + Z_1, V_5, Z_3 \rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_5$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_4 + Z_1 &= [\alpha_1, 0, 0, \sqrt{1 + c^2(x^2+y^2+z^2)}, \beta_1 u], \\ V_5 &= [0, 0, \sqrt{1 + c^2(x^2+y^2+z^2)} + cz, -cy, 0], \\ Z_3 &= [1, 0, 0, 0, \alpha_3 u] \end{aligned} \tag{3.69}$$

The transversality condition (1.43) of (3.69) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_5$ is $\langle V_5 \rangle$. Therefore, one can find the invariants of $\mathcal{L}_5$ by starting with V_5. The invariants of V_5 are:

$$m_1 = t, m_2 = x, m_3 = u, m_4 = \frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2(x^2 + y^2)} \tag{3.70}$$

The operators can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$\begin{aligned} V_4 + Z_1 &= [\alpha_1, 0, \beta_1 m_3, -c m_4], \\ Z_3 &= [1, 0, \alpha_3 m_3, 0] \end{aligned} \tag{3.71}$$

Next, the invariants of $V_4 + Z_1$ are

$$n_1 = m_2, n_2 = m_3 \mathrm{e}^{-\frac{\beta_1}{\alpha_1} m_1}, n_3 = m_4 \mathrm{e}^{\frac{c}{\alpha_1} m_1} \tag{3.72}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$Z_3 = [0, \tfrac{(\alpha_3 \alpha_1 - \beta_1)}{\alpha_1} n_2, \tfrac{c}{\alpha_1} n_3] \tag{3.73}$$

Finally, the invariants of Z_3 are

$$n_1, n_2 \; n_3^{\frac{\beta_1 - \alpha_3 \alpha_1}{c}} \tag{3.74}$$

Writing the invariants (3.74) in terms of the original variables gives the joint invariants of $\mathcal{L}_5$ as

$$x, u \; e^{-\alpha_3 t} \Big(\frac{1 + c^2(x^2 + y^2)}{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}} \Big)^{A_1} \tag{3.75}$$

where $A_1 = \frac{\beta_1 - \alpha_1 \alpha_3}{c}$.

Therefore, the invariant transformations are:

$$w = x,\ Z(w) = u\ e^{-\alpha_3 t}\Big(\frac{1 + c^2(x^2 + y^2)}{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}\Big)^{A_1} \tag{3.76}$$

Thus, using (3.76), equation (3.3) can be reduced to the ODE:

$$\left(c^2 w^2 + 1\right) Z'' - c^2 \left(2A_1 - 3\right) w Z' + \left(c^2 (A_1^2 - 2A_1) - \alpha_3^2\right) Z = 0 \tag{3.77}$$

It was found using Maple software that the transformation

$$w = \frac{r}{ic},\ Z\left(w\right) = \left(r^2 - 1\right)^{\frac{1}{2}A_1 - \frac{1}{4}} R\left(r\right) \tag{3.78}$$

reduces equation (3.77) to the associated Legendre equation

$$(1 - r^2)R'' - 2rR' + \Big(\nu(\nu + 1) - \frac{\mu^2}{1 - r^2}\Big)R = 0 \tag{3.79}$$

with $\nu = \frac{2\sqrt{c^2 + \alpha_3^2} - c}{2c}$, $\mu = A_1 - \frac{1}{2}$. Therefore, the solution of (3.77) is

$$Z\left(w\right) = \left(-c^2 w^2 - 1\right)^{\frac{\mu}{2}} \Big(c_1 \mathrm{P}_\nu^\mu(icw) + c_2 \mathrm{Q}_\nu^\mu(icw)\Big) \tag{3.80}$$

where P_ν^μ and Q_ν^μ are the associated Legendre functions of the first and second kinds respectively. Thus, the invariant solution of (3.3) is

$$u(t,x,y,z) = Z(x)\ e^{\alpha_3 t}\Big(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}\Big)^{\mu+\frac{1}{2}} \tag{3.81}$$

where $\alpha_3^2 = c^2(\nu+\frac{1}{2})^2 - c^2$.

3.7.1.6 Case $\mathcal{L}_6 = \langle V_5+Z_1, V_6+Z_2, Z_3\rangle$, $Z_3 = V_7+\alpha_3 V_8$

The generators of $\mathcal{L}_6$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_5+Z_1 &= [\alpha_1, 0, \sqrt{1+c^2(x^2+y^2+z^2)}+cz, -cy, \beta_1 u], \\ V_6+Z_2 &= [\alpha_2, \sqrt{1+c^2(x^2+y^2+z^2)}+cz, 0, -cx, \beta_2\ u], \\ Z_3 &= [1,0,0,0,\alpha_3\ u] \end{aligned} \tag{3.82}$$

The transversality condition (1.43) of (3.82) with rank three is always satisfied. Since the Lie algebra generated by $\mathcal{L}_6$ is abelian, one can find the invariants of $\mathcal{L}_6$ in any order. The invariants of Z_3 are:

$$m_1 = x, m_2 = y, m_3 = z, m_4 = u\mathrm{e}^{-\alpha_3 t} \tag{3.83}$$

The operators can be given in terms of the variables $m_i, i = 1,\ldots,4$ as

$$\begin{aligned} V_5+Z_1 &= [0, \sqrt{1+c^2\,({m_1}^2+{m_2}^2+{m_3}^2)}+cm_3, -cm_2, (\beta_1-\alpha_3\,\alpha_1)m_4], \\ V_6+Z_2 &= [\sqrt{1+c^2\,({m_1}^2+{m_2}^2+{m_3}^2)}+c\ m_3, 0, -c\ m_1, (\beta_2-\alpha_2\,\alpha_3)\,m_4] \end{aligned} \tag{3.84}$$

Next, the invariants of $V_5 + Z_1$ are

$$
\begin{aligned}
n_1 &= m_1, \\
n_2 &= \frac{-cm_3+\sqrt{1+c^2({m_1}^2+{m_2}^2+{m_3}^2)}}{1+c^2({m_1}^2+{m_2}^2)}, \\
n_3 &= m_4 \mathrm{e}^{\frac{(\alpha_3\,\alpha_1-\beta_1)m_2\left(-cm_3+\sqrt{1+c^2({m_1}^2+{m_2}^2+{m_3}^2)}\right)}{1+c^2({m_1}^2+{m_2}^2)}}.
\end{aligned}
\tag{3.85}
$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$
V_6 + Z_2 = [\tfrac{1}{n_2}, 0, (\beta_2 - \alpha_2\,\alpha_3)\,n_3] \tag{3.86}
$$

Finally, the invariants of $V_6 + Z_2$ are

$$
n_2, n_3\ \mathrm{e}^{(\alpha_2\,\alpha_3-\beta_2)n_1 n_2} \tag{3.87}
$$

Writing the invariants (3.87) in terms of the original variables gives the joint invariants of $\mathcal{L}_6$ as

$$
\frac{-cz+\sqrt{1+c^2\,(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}, ue^{-\alpha t}e^{\left(A_1x+A_2y\right)\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}\right)} \tag{3.88}
$$

where $A_1 = \alpha_2\alpha_3 - \beta_2,\ A_2 = \alpha_1\alpha_3 - \beta_1$.

Therefore, the invariant transformations are:

$$
w = \frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)},\ Z(w) = ue^{-\alpha t}e^{\left(A_1x+A_2y\right)\left(\frac{-cz+\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}\right)} \tag{3.89}
$$

Thus, using (3.89), equation (3.3) can be reduced to the ODE:

$$c^2w^2Z'' - c^2wZ' + \left((A_1^2 + A_2^2)w^2 - \alpha_3^2\right)Z = 0 \tag{3.90}$$

which can be transformed using the transformation $w = r,\ Z(w) = rR(r)$ to the parametric Bessel equation:

$$r^2R'' + rR' + \left(\alpha^2r^2 - v^2\right)R = 0 \tag{3.91}$$

with $\alpha = \frac{\sqrt{{A_1}^2+{A_2}^2}}{c},\ v = \frac{\sqrt{c^2+\alpha^2}}{c}$.

Therefore the solution of (3.90) is

$$Z(w) = c_1w\mathrm{J}_v(\alpha w) + c_2w\mathrm{Y}_v(\alpha w) \tag{3.92}$$

where J_v and Y_v are the Bessel functions of the first and second kind respectively. Thus, the invariant solution of (3.3) is

$$u(t,x,y,z) = Z\left(\frac{-cz + \sqrt{1 + c^2(x^2 + y^2 + z^2)}}{1 + c^2(x^2 + y^2)}\right) e^{\alpha_3 t} e^{\left(A_1x+A_2y\right)\left(\frac{cz-\sqrt{1+c^2(x^2+y^2+z^2)}}{1+c^2(x^2+y^2)}\right)} \tag{3.93}$$

3.7.2 Non-solvable subalgebras of $\mathcal{L}$

3.7.2.1 Case $\mathcal{L}_1 = \langle V_4, V_5, V_5 - 2cV_2 \rangle$

Since the reduced row echelon form of the operators of $\mathcal{L}_1$ coincides with (2.26), the same argument applies.

3.8 Invariants solutions in spherical coordinates

In this section, we summarize all the obtained invariant solutions of rank one for the wave equation on Anti-Einstein spacetime (3.2) as follows:

I) Solution (3.26):

$$u\,(t, r, \theta) = \mathrm{e}^{\frac{\beta_1}{\alpha_1} t} Z\left(\frac{\mathrm{e}^{ct}\left(\sqrt{c^2 r^2+1} - cr\cos\theta\right)}{1+c^2r^2\sin^2\theta}\right),\ \alpha_1 \neq 0, \tag{3.94}$$

where $Z(w)$ is given according to the following cases:

i) $\alpha^2 - \alpha_1^2 \neq 0$:

$$Z\,(w) = c_1\, w^{\frac{-c{\alpha_1}^2 - \alpha\,\beta_1 + \alpha_1\sqrt{c^2{\alpha_1}^2 + 2\,c\alpha\,\beta_1 + {\beta_1}^2}}{(\alpha^2 - {\alpha_1}^2)c}} + c_2\, w^{\frac{-c{\alpha_1}^2 - \alpha\,\beta_1 - \alpha_1\sqrt{c^2{\alpha_1}^2 + 2\,c\alpha\,\beta_1 + {\beta_1}^2}}{(\alpha^2 - {\alpha_1}^2)c}} \tag{3.95}$$

ii) $\alpha^2 - \alpha_1^2 = 0,\ c\alpha_1 + \beta_1 \neq 0$:

$$Z\,(w) = c_1\, w^{-\frac{\beta_1^2}{2c\alpha_1\,(\alpha_1\,c + \beta_1)}} \tag{3.96}$$

II) Solution (3.31):

$$u\left(t,r,\theta\right)=Z\left(t\right)\left(\frac{1+c^2r^2\sin^2\theta}{\sqrt{c^2r^2+1}-cr\cos\theta}\right)^{\frac{\beta_1}{\alpha\,c}},\ \ \alpha\neq 0, \tag{3.97}$$

where

$$Z\left(w\right)=c_1\,\mathrm{e}^{\frac{\sqrt{\beta_1^2+2\beta_1\,\alpha\,c}}{\alpha}w}+c_2\,\mathrm{e}^{-\frac{\sqrt{\beta_1^2+2\beta_1\,\alpha\,c}}{\alpha}w} \tag{3.98}$$

III) Solution (3.39):

$$u\left(t,r,\theta\right)=\mathrm{e}^{\frac{\beta_1}{\alpha_1}t}Z\left(\frac{\left(\sqrt{c^2r^2+1}-cr\cos\theta\right)}{1+c^2r^2\sin^2\theta}\right),\ \ \alpha_1\neq 0 \tag{3.99}$$

where

$$Z\left(w\right)=c_1\,w^{\frac{c\alpha_1+\sqrt{c^2{\alpha_1}^2+{\beta_1}^2}}{c\alpha_1}}+c_2\,w^{\frac{c\alpha_1-\sqrt{c^2{\alpha_1}^2+{\beta_1}^2}}{c\alpha_1}} \tag{3.100}$$

IV) Solution (3.49):

$$u\left(t,r,\theta\right)=\mathrm{e}^{\beta_1\,t}Z\left(\frac{\mathrm{e}^{ct}\left(\sqrt{c^2r^2+1}-cr\cos\theta\right)}{1+c^2r^2\sin^2\theta}\right) \tag{3.101}$$

where $Z(w)$ is given according to the following cases:

i) $\alpha_1^2-1\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{c{\alpha_1}^2+\beta_1+\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\beta_1+{\beta_1}^2}}{c\left({\alpha_1}^2-1\right)}}+c_2\,w^{\frac{c{\alpha_1}^2+\beta_1-\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\beta_1+{\beta_1}^2}}{c\left({\alpha_1}^2-1\right)}} \tag{3.102}$$

ii) $\alpha_1^2 - 1 = 0,\ c + \beta_1 \neq 0$:

$$Z\,(w) = c_1\, w^{-\,\frac{\beta_1^2}{2c\,(\,c+\beta_1)}} \tag{3.103}$$

V) Solution (3.55):

$$u\,(t,r,\theta) = Z\,(t)\left(\frac{1+c^2r^2\sin^2\theta}{\sqrt{c^2r^2+1}-cr\cos\theta}\right)^{\frac{\beta_1}{c}} \tag{3.104}$$

where

$$Z\,(w) = c_1\, \mathrm{e}^{\sqrt{\beta_1^2+2\,c\beta_1}w} + c_2\, \mathrm{e}^{-\sqrt{\beta_1^2+2\,c\beta_1}w} \tag{3.105}$$

VI) Solution (3.68):

$$u\,(t,r,\theta,\varphi) = \mathrm{e}^{\alpha_3\,t}\mathrm{e}^{i(\gamma-1)\arctan(\cot\varphi)}\left(cr\cos\theta + \sqrt{1+c^2r^2}\right)^{3\nu-\gamma-\mu} Z\left(r^2\sin^2\theta\right) \tag{3.106}$$

where

$$Z(w) = (-c^2w)^{\frac{1}{2}\,(\gamma-1)}$$
$$\Big(c_1F(\nu,\mu;\gamma;-c^2w) + c_2(-c^2w)^{1-\gamma}F(\nu-\gamma+1,\mu-\gamma+1;2-\gamma;-c^2w)\Big), \tag{3.107}$$

$F(\nu,\mu;\gamma;r)$ is the hypergeometric function and $\alpha_3^2 = c^2(\nu-\mu)^2 - c^2$.

VII) Solution (3.81):

$$u\,(t,r,\theta,\varphi) = \mathrm{e}^{\alpha\,t}\left(\frac{\sqrt{c^2r^2+1}-cr\cos\theta}{1+c^2r^2\sin^2\theta}\right)^{A_1} Z\,(r\cos\varphi\sin\theta) \tag{3.108}$$

where

$$Z(w) = \left(-c^2w^2 - 1\right)^{\frac{\mu}{2}} \left(c_1 \mathrm{P}_\nu^\mu(icw) + c_2 \mathrm{Q}_\nu^\mu(icw) \right), \tag{3.109}$$

P_ν^μ and Q_ν^μ are the associated Legendre functions of the first and second kind respectively.

VIII) Solution (3.93):

$$\begin{aligned} u(t, r, \theta, \varphi) = {} & \mathrm{e}^{\alpha_3 (1 + c^2 r^2 \sin^2\theta) t} \mathrm{e}^{r \sin\theta (cr\cos\theta - \sqrt{1+c^2r^2})(A_1 \cos\varphi + A_2 \sin\varphi)} \\ & Z\left(\tfrac{\sqrt{c^2r^2+1} - cr\cos\theta}{1 + c^2 r^2 \sin^2\theta} \right) \end{aligned} \tag{3.110}$$

where

$$Z(w) = c_1 w \mathrm{J}_v(\alpha w) + c_2 w \mathrm{Y}_v(\alpha w), \tag{3.111}$$

J_v and Y_v are the Bessel functions of the first and second kind respectively.

IX) The solution that comes from the non-solvable algebra is

$$u(t, r, \theta, \varphi) = c_1 + c_2\, t \tag{3.112}$$

CHAPTER 4

WAVE EQUATION ON NON-STATIC ANALOG OF EINSTEIN SPACETIME

In this chapter, we will discuss the wave equation $\Delta u = 0$ on the spherically symmetric space time with the metric

$$ds^2 = (c^2t^2 - 1)^{-1}dt^2 - dr^2 - t^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.1}$$

Hence, the corresponding wave equation on this metric is

$$\begin{aligned} &\frac{\partial}{\partial t}\left(t^2 u_t \sin\theta\sqrt{c^2t^2-1}\right) - \frac{\partial}{\partial r}\left(\frac{t^2 u_r \sin\theta}{\sqrt{c^2t^2-1}}\right) - \frac{\partial}{\partial \theta}\left(\frac{u_\theta \sin\theta}{\sqrt{c^2t^2-1}}\right) - \\ &\frac{\partial}{\partial \varphi}\left(\frac{u_\varphi}{\sin\theta\sqrt{c^2t^2-1}}\right) = 0. \end{aligned} \tag{4.2}$$

Equation (4.2) can be written in Cartesian coordinates as:

$$\begin{aligned}
&t^2\left(x^2+y^2+z^2\right)\left(c^2t^2-1\right)u_{tt}-\Big(\left(y^2+z^2\right)^2+x^2\left(y^2+z^2+t^2\right)\Big)\,u_{xx}\\
&-\Big(\left(x^2+z^2\right)^2+y^2\left(x^2+z^2+t^2\right)\Big)\,u_{yy}-\Big(\left(x^2+y^2\right)^2+z^2\left(x^2+y^2+t^2\right)\Big)\,u_{zz}\\
&-2\left(t^2-x^2-y^2-z^2\right)\Big(xy\;u_{xy}+\,xz\;u_{xz}+\,yz\;u_{yz}\Big)\\
&+\left(x^2+y^2+z^2\right)\Big(\left(3c^2t^3-2\,t\right)u_t+2\,x\;u_x+2\,y\;u_y+2\,z\;u_z\Big)=0
\end{aligned} \tag{4.3}$$

4.1 Lie point symmetries of the wave equation on non-static analog of Einstein spacetime

By using Theorem 1.31 and the isometries of the metric (2.1) given in [21], the Lie symmetry algebra of the wave equation (4.2) consists of the eight-dimensional subalgebra spanned by

$$\begin{aligned}
X_1&=[A\sin\varphi\sin\theta,0,\tfrac{A\sin\varphi\cos\theta}{t},\tfrac{A\cos\varphi}{t\sin\theta},0]\\
X_2&=[A\cos\varphi\sin\theta,0,\tfrac{A\cos\varphi\cos\theta}{t},-\tfrac{A\sin\varphi}{t\sin\theta},0]\\
X_3&=[A\cos\theta,0,-\tfrac{A\sin\theta}{t},0,0]\\
X_4&=[0,0,-\cos\varphi,\tfrac{\sin\varphi}{\tan\theta},0]\\
X_5&=[0,0,\sin\varphi,\tfrac{\cos\varphi}{\tan\theta},0]\\
X_6&=[0,0,0,1,0]\\
X_7&=[0,1,0,0,0]\\
X_8&=[0,0,0,0,u]
\end{aligned} \tag{4.4}$$

and the infinite-dimensional ideal consisting of the operators

$$X_\tau = \tau(t, r, \theta, \varphi)\frac{\partial}{\partial u} \tag{4.5}$$

where $\tau(t, r, \theta, \varphi)$ is an arbitrary solution of the wave equation (4.2)

Moreover, the generators (4.4) can be written in Cartesian coordinates as follows:

$$\begin{aligned}
X_1 &= [\tfrac{Ay}{B}, -\tfrac{Ayx}{Bt}, \tfrac{A\left(x^2+z^2\right)}{Bt}, -\tfrac{Ayz}{Bt}, 0] \\
X_2 &= [\tfrac{Ax}{B}, \tfrac{A\left(y^2+z^2\right)}{Bt}, -\tfrac{Ayx}{Bt}, -\tfrac{Azx}{Bt}, 0] \\
X_3 &= [\tfrac{Az}{B}, -\tfrac{Azx}{Bt}, -\tfrac{Ayz}{Bt}, \tfrac{A\left(x^2+y^2\right)}{Bt}, 0] \\
X_4 &= [0, -z, 0, x, 0] \\
X_5 &= [0, 0, z, -y, 0] \\
X_6 &= [0, -y, x, 0, 0] \\
X_7 &= [0, \tfrac{x}{B}, \tfrac{y}{B}, \tfrac{z}{B}, 0] \\
X_8 &= [0, 0, 0, 0, u].
\end{aligned} \tag{4.6}$$

where $A = \sqrt{c^2t^2 - 1}$, $B = \sqrt{x^2 + y^2 + z^2}$.

4.2 Lie point symmetry transformations

The one-parameter groups $G_i(\varepsilon) = \{e^{\varepsilon X_i} : \varepsilon \in \mathbb{R}\}$ generated by (4.6) are given by solving the initial value problems (1.5) as follows:

$$
\begin{aligned}
&G_1(\varepsilon_1) : (t,x,y,z,u) \mapsto (K(y,\varepsilon_1), \tfrac{xt}{K(y,\varepsilon_1)}, \tfrac{cty\cosh c\varepsilon_1 + AB\sinh c\varepsilon_1}{cK(y,\varepsilon_1)}, \tfrac{zt}{K(y,\varepsilon_1)}, u)\\
&G_2(\varepsilon_2) : (t,x,y,z,u) \mapsto (K(x,\varepsilon_2), \tfrac{ctx\cosh c\varepsilon_2 + AB\sinh c\varepsilon_2}{cK(x,\varepsilon_2)}, \tfrac{yt}{K(x,\varepsilon_2)}, \tfrac{zt}{K(x,\varepsilon_2)}, u)\\
&G_3(\varepsilon_3) : (t,x,y,z,u) \mapsto (K(z,\varepsilon_3), \tfrac{xt}{K(z,\varepsilon_3)}, \tfrac{yt}{K(z,\varepsilon_3)}, \tfrac{ctz\cosh c\varepsilon_3 + AB\sinh c\varepsilon_3}{cK(z,\varepsilon_3)}, u)\\
&G_4(\varepsilon_4) : (t,x,y,z,u) \mapsto (t, x\sin\varepsilon_4 - z\cos\varepsilon_4, y, x\sin\varepsilon_4 + z\cos\varepsilon_4, u)\\
&G_5(\varepsilon_5) : (t,x,y,z,u) \mapsto (t, x, -z\sin\varepsilon_5 - y\cos\varepsilon_5, y\sin\varepsilon_5 - z\cos\varepsilon_5, u)\\
&G_6(\varepsilon_6) : (t,x,y,z,u) \mapsto (t, -y\sin\varepsilon_6 + x\cos\varepsilon_5, x\sin\varepsilon_6 + y\cos\varepsilon_6, z, u)\\
&G_7(\varepsilon_7) : (t,x,y,z,u) \mapsto (t, \tfrac{(B+\varepsilon_7)}{B}x, \tfrac{(B+\varepsilon_7)}{B}y, \tfrac{(B+\varepsilon_7)}{B}z, u)\\
&G_8(\varepsilon_8) : (t,x,y,z,u) \mapsto (t, x, y, z, u+\varepsilon_8)
\end{aligned}
\tag{4.7}
$$

where

$$K(\gamma,\varepsilon) = \tfrac{1}{Bc}\sqrt{\left(ct\gamma\sinh c\varepsilon + AB\cosh c\varepsilon\right)^2 + B^2}.$$

4.3 Lie Algebra Structure

The non-zero Lie brackets of (4.4) are:

$$
\begin{array}{llll}
[X_1,X_2] = -c^2X_6, & [X_1,X_3] = -c^2X_5, & [X_1,X_5] = -X_3, & [X_1,X_6] = -X_2\\
[X_2,X_3] = c^2X_4, & [X_2,X_4] = X_3, & [X_2,X_6] = X_1, & [X_3,X_4] = -X_2,\\
[X_3,X_5] = X_1, & [X_4,X_5] = X_6, & [X_4,X_6] = -X_5, & [X_5,X_6] = X_4.
\end{array}
\tag{4.8}
$$

It is clear that the Lie brackets above are exactly the same as the Lie brackets of Anti-Einstein spacetime. Therefore, the Lie algebra $\mathcal{L} = so(1,3) \oplus \mathbb{R}^2$ and the optimal systems can be given using Theorems 3.2 and 3.3.

4.4 Joint invariants and invariant solutions of three-dimensional optimal systems of $\mathcal{L} = so(1,3) \oplus \mathbb{R}^2$

4.4.1 Solvable subalgebras of $\mathcal{L}$

4.4.1.1 Case $\mathcal{L}_1 = \langle V_3 + \alpha V_4 + Z_1, V_5, V_6 \rangle, \alpha \neq 0$

The generators of $\mathcal{L}_1$ in Cartesian coordinates are as follows:

$$
\begin{aligned}
V_3 + \alpha V_4 + Z_1 &= [\tfrac{\alpha z\ A}{B}, \tfrac{-\alpha\, xzA + \alpha_1 x\, t - ytB}{tB}, \tfrac{xtB + \alpha_1\, ty - \alpha\, yzA}{tB}, \tfrac{\alpha\left(x^2+y^2\right)A + \alpha_1\, tz}{tB}, \beta_1 u], \\
V_5 &= [\tfrac{yA}{B}, -\tfrac{yxA}{tB}, \tfrac{\left(x^2+z^2\right)A + cztB}{tB}, -\tfrac{y(zA+ctB)}{tB}, 0], \\
V_6 &= [\tfrac{xA}{B}, \tfrac{\left(y^2+z^2\right)A + cztB}{tB}, -\tfrac{yxA}{tB}, -\tfrac{x(zA+ctB)}{tB}, 0]
\end{aligned}
\tag{4.9}
$$

where $A = \sqrt{c^2t^2 - 1}$, $\quad B = \sqrt{x^2 + y^2 + z^2}$.

The transversality condition (1.43) of (4.9) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_1$ is $\langle V_5, V_6 \rangle$ which is abelian. Therefore, one can find the invariants of $\mathcal{L}_1$ by starting with V_5 or V_6. The invariants of V_5 are:

$$m_1 = xt, m_2 = \sqrt{x^2+y^2+z^2}, m_3 = ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}, m_4 = u \tag{4.10}$$

The operator V_6 can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$V_6 = [m_3, 0, 0, 0] \tag{4.11}$$

Next, the invariants of V_6 are

$$n_1 = m_2, n_2 = m_3, n_3 = m_4 \tag{4.12}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_3 + \alpha V_4 + Z_1 = [\alpha_1, \tfrac{n_2(\alpha c\ n_1+\alpha_1)}{n_1}, \beta_1 n_3] \tag{4.13}$$

Now, we have to study the following two cases:

- <u>Case 1</u>: If $\alpha_1 \neq 0$, the invariants of $V_3 + \alpha V_4 + Z_1$ are

$$\frac{n_2}{n_1}\mathrm{e}^{-\frac{\alpha c}{\alpha_1}n_1}, n_3\mathrm{e}^{-\frac{\beta_1}{\alpha_1}n_1} \tag{4.14}$$

Writing the invariants (4.14) in terms of the original variables gives the joint

invariants of $\mathcal{L}_1$ as

$$\left(\frac{ctz}{\sqrt{x^2+y^2+z^2}}+\sqrt{c^2t^2-1}\right)\mathrm{e}^{-\frac{\alpha\, c\sqrt{x^2+y^2+z^2}}{\alpha_1}},u\ \mathrm{e}^{-\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.15}$$

Therefore, the invariant transformations are:

$$w=\left(\frac{ctz}{\sqrt{x^2+y^2+z^2}}+\sqrt{c^2t^2-1}\right)\mathrm{e}^{-\frac{\alpha\, c\sqrt{x^2+y^2+z^2}}{\alpha_1}},\ Z(w)=u\ \mathrm{e}^{-\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.16}$$

Thus, using (4.16), equation (4.3) can be reduced to the ODE:

$$c^2w^2\left(\alpha^2-{\alpha_1}^2\right)Z''+c\Big(\left(\alpha^2-3\,{\alpha_1}^2\right)c-2\,\beta_1\,\alpha\Big)wZ'+{\beta_1}^2Z=0 \tag{4.17}$$

which has the non-trivial solution for the following cases:

i) $\alpha^2-\alpha_1^2\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{c{\alpha_1}^2+\alpha\,\beta_1+\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{c\left(\alpha^2-{\alpha_1}^2\right)}}+c_2\,w^{\frac{c{\alpha_1}^2+\alpha\,\beta_1-\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{c\left(\alpha^2-{\alpha_1}^2\right)}} \tag{4.18}$$

ii) $\alpha^2-\alpha_1^2=0, c\alpha+\beta_1\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{\beta_1^2}{2c\alpha\,(c\alpha+\beta_1)}} \tag{4.19}$$

Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\Big(\big(\frac{ctz}{\sqrt{x^2+y^2+z^2}} + \sqrt{c^2t^2-1}\big)\mathrm{e}^{-\frac{\alpha\,c\sqrt{x^2+y^2+z^2}}{\alpha_1}}\Big)\mathrm{e}^{\frac{\beta_1\,\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.20}$$

- Case 2: If $\alpha_1 = 0$, the invariants of $V_3 + \alpha V_4 + Z_1$ are

$$n_1, n_3 n_2^{-\frac{\beta_1}{\alpha\,c}} \tag{4.21}$$

Writing the invariants (4.21) in terms of the original variables gives the joint invariants of $\mathcal{L}_1$ as

$$\sqrt{x^2+y^2+z^2}, u\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{-A_1}$$

where $A_1 = \frac{\beta_1}{\alpha\,c}$. Therefore, the invariant transformations are:

$$w = \sqrt{x^2+y^2+z^2},\ Z(w) = u\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{-A_1} \tag{4.22}$$

Thus, using (4.22), equation (4.3) can be reduced to the ODE:

$$w^2 Z'' + 2A_1\, wZ' - A_1\big(1 - A_1 + c^2(2+A_1)w^2\big)Z = 0 \tag{4.23}$$

which can be transformed using the transformation

$w = r, Z(w) = r^{-A_1}\ R(r)$ to the following linear second order ODE with constant coefficients

$$R'' - c^2 A_1(A_1+2)R = 0 \tag{4.24}$$

Therefore, the solution of (4.23) is

$$Z\left(w\right) = w^{-A_1}\left(c_1\,\mathrm{e}^{c\sqrt{2A_1+A_1^2}w} + c_2\,\mathrm{e}^{-c\sqrt{2A_1+A_1^2}w}\right) \tag{4.25}$$

Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\left(\sqrt{x^2+y^2+z^2}\right)\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{A_1} \tag{4.26}$$

4.4.1.2 Case $\mathcal{L}_2 = \langle V_3 + Z_1, V_5, V_6\rangle$

The generators of $\mathcal{L}_2$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_3 + Z_1 &= [0, \tfrac{\alpha_1\, x - yB}{B}, \tfrac{\alpha_1\, y + xB}{B}, \tfrac{\alpha_1\, z}{B}, \beta_1 u], \\ V_5 &= [\tfrac{yA}{B}, -\tfrac{yxA}{tB}, \tfrac{\left(x^2+z^2\right)A + cztB}{tB}, -\tfrac{y(zA+ctB)}{tB}, 0], \\ V_6 &= [\tfrac{xA}{B}, \tfrac{\left(y^2+z^2\right)A + cztB}{tB}, -\tfrac{yxA}{tB}, -\tfrac{x(zA+ctB)}{tB}, 0] \end{aligned} \tag{4.27}$$

where $A = \sqrt{c^2t^2-1}$, $\;B = \sqrt{x^2+y^2+z^2}$.

The transversality condition (1.43) of (4.27) with rank three is satisfied for $\alpha_1 \neq 0$. Since the derived Lie algebra generated by $\mathcal{L}_2$ is $\langle V_5, V_6\rangle$. Therefore, one can find the invariants of $\mathcal{L}_2$ by starting with the invariants of $\langle V_5, V_6\rangle$ which are given by (4.12) as

$$n_1 = \sqrt{x^2+y^2+z^2}, n_2 = ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}, n_3 = u \tag{4.28}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_3 + Z_1 = [\alpha_1, \tfrac{\alpha_1}{n_1} n_2, \beta_1 n_3] \tag{4.29}$$

Finally, the invariants of $V_3 + Z_1$ are

$$\frac{n_2}{n_1}, n_3 \mathrm{e}^{-\frac{\beta_1}{\alpha_1} n_1} \tag{4.30}$$

Writing the invariants (4.30) in terms of the original variables gives the joint invariants of $\mathcal{L}_2$ as

$$\frac{czt + \sqrt{x^2+y^2+z^2}\sqrt{c^2t^2-1}}{\sqrt{x^2+y^2+z^2}}, u\mathrm{e}^{-\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.31}$$

Therefore, the invariant transformations are:

$$w = \frac{czt + \sqrt{x^2+y^2+z^2}\sqrt{c^2t^2-1}}{\sqrt{x^2+y^2+z^2}},\ Z(w) = u\mathrm{e}^{-\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.32}$$

Thus, using (4.32), equation (4.3) can be reduced to the ODE:

$$\alpha_1{}^2 c^2 w^2 Z'' + 3\alpha_1{}^2 c^2 w Z' - \beta_1^2 Z = 0 \tag{4.33}$$

which has the solution

$$Z(w) = c_1\, w^{\frac{-c\alpha_1+\sqrt{\alpha_1^2c^2+\beta_1^2}}{\alpha_1 c}} + c_2\, w^{\frac{-c\alpha_1-\sqrt{\alpha_1^2c^2+\beta_1^2}}{\alpha_1 c}} \tag{4.34}$$

Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\Big(\frac{czt+\sqrt{x^2+y^2+z^2}\sqrt{c^2t^2-1}}{\sqrt{x^2+y^2+z^2}}\Big)\mathrm{e}^{\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.35}$$

4.4.1.3 Case $\mathcal{L}_3 = \langle V_4 + Z_1, V_5, V_6\rangle$

The generators of $\mathcal{L}_3$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_4 + Z_1 &= [\tfrac{zA}{B}, \tfrac{x(\alpha_1 t - zA)}{tB}, \tfrac{y(\alpha_1 t - zA)}{tB}, \tfrac{(x^2+y^2)A+\alpha_1 tz}{tB}, \beta_1 u], \\ V_5 &= [\tfrac{yA}{B}, -\tfrac{yxA}{tB}, \tfrac{(x^2+z^2)A+cztB}{tB}, -\tfrac{y(zA+ctB)}{tB}, 0], \\ V_6 &= [\tfrac{xA}{B}, \tfrac{(y^2+z^2)A+cztB}{tB}, -\tfrac{yxA}{tB}, -\tfrac{x(zA+ctB)}{tB}, 0] \end{aligned} \tag{4.36}$$

where $A = \sqrt{c^2t^2-1}$, $B = \sqrt{x^2+y^2+z^2}$.

The transversality condition (1.43) of (4.36) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_3$ is $\langle V_5, V_6\rangle$. Therefore, one can find the invariants of $\mathcal{L}_3$ by starting with the invariants of $\langle V_5, V_6\rangle$ which are given by (4.12) as

$$n_1 = \sqrt{x^2+y^2+z^2}, n_2 = ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}, n_3 = u \tag{4.37}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_4 + Z_1 = [\alpha_1, \tfrac{n_2(\alpha_1 + cn_1)}{n_1}, \beta_1 n_3] \tag{4.38}$$

Now, we have to study the following two cases:

- Case 1: If $\alpha_1 \neq 0$, the invariants of $V_4 + Z_1$ are

$$\frac{n_2}{n_1} e^{-\frac{c}{\alpha_1} n_1}, n_3 e^{-\frac{\beta_1}{\alpha_1} n_1} \tag{4.39}$$

Writing the invariants (4.39) in terms of the original variables gives the joint invariants of $\mathcal{L}_3$ as

$$\left(\frac{ctz}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{c^2 t^2 - 1} \right) e^{-\frac{\alpha\, c \sqrt{x^2+y^2+z^2}}{\alpha_1}}, u\ e^{-\frac{\beta_1 \sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.40}$$

Therefore, the invariant transformations are:

$$w = \left(\frac{ctz}{\sqrt{x^2 + y^2 + z^2}} + \sqrt{c^2 t^2 - 1} \right) e^{-\frac{c \sqrt{x^2+y^2+z^2}}{\alpha_1}},\ Z(w) = u\ e^{-\frac{\beta_1 \sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.41}$$

Thus, using (4.41), equation (4.3) can be reduced to the ODE:

$$c^2 \left(\alpha_1^2 - 1 \right) w^2 Z'' + c\Big(\left(3{\alpha_1}^2 - 1 \right) c + 2\, \beta_1 \Big) w Z' - {\beta_1}^2 Z = 0 \tag{4.42}$$

which has the non-trivial solution for the following cases:

i) $\alpha_1^2 - 1 \neq 0$:

$$Z(w) = c_1\, w^{\frac{-c{\alpha_1}^2 - \beta_1 + \alpha_1 \sqrt{c^2{\alpha_1}^2 + 2\,c\,\beta_1 + {\beta_1}^2}}{c({\alpha_1}^2 - 1)}} + c_2\, w^{\frac{-c{\alpha_1}^2 - \beta_1 - \alpha_1 \sqrt{c^2{\alpha_1}^2 + 2\,c\,\beta_1 + {\beta_1}^2}}{c({\alpha_1}^2 - 1)}} \tag{4.43}$$

ii) $\alpha_1^2 - 1 = 0, c + \beta_1 \neq 0$:

$$Z(w) = c_1\, w^{\frac{\beta_1^2}{2c\,(c+\beta_1)}} \tag{4.44}$$

Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\Big(\big(\frac{ctz}{\sqrt{x^2+y^2+z^2}} + \sqrt{c^2t^2-1}\big)\mathrm{e}^{-\frac{c\sqrt{x^2+y^2+z^2}}{\alpha_1}}\Big)\mathrm{e}^{\frac{\beta_1\sqrt{x^2+y^2+z^2}}{\alpha_1}} \tag{4.45}$$

- Case 2: If $\alpha_1 = 0$, the invariants of $V_4 + Z_1$ are

$$n_1, n_3 n_2^{-\frac{\beta_1}{c}} \tag{4.46}$$

Writing the invariants (4.46) in terms of the original variables gives the joint invariants of $\mathcal{L}_1$ as

$$\sqrt{x^2+y^2+z^2}, u\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{-\frac{\beta_1}{c}}$$

Therefore, the invariant transformations are:

$$w = \sqrt{x^2+y^2+z^2},\; Z(w) = u\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{-\frac{\beta_1}{c}} \tag{4.47}$$

Thus, using (4.47), equation (4.3) can be reduced to the ODE:

$$c^2w^2Z'' + 2c\beta_1\, wZ' - \beta_1\, \left(c - \beta_1 + c^2(2\,c + \beta_1\,)w^2\right) Z = 0 \tag{4.48}$$

which can be transformed using the transformation $w = r, Z(w) = r^{-\frac{\beta_1}{c}}\, R(r)$ to the following linear second order ODE with constant coefficients

$$R'' - \beta_1(\beta_1 + 2c)R = 0 \tag{4.49}$$

Therefore, the solution of (4.48) is

$$Z\,(w) = w^{-\frac{\beta_1}{c}} \left(c_1\, \mathrm{e}^{\sqrt{\beta_1{}^2+2\,\beta_1 c}\;\, w} + c_2\, \mathrm{e}^{-\sqrt{\beta_1{}^2+2\,\beta_1 c}\;\, w}\right) \tag{4.50}$$

Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\big(\sqrt{x^2+y^2+z^2}\big)\left(ctz + \sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}\right)^{\frac{\beta_1}{c}} \tag{4.51}$$

4.4.1.4 Case $\mathcal{L}_4 = \langle V_3 + Z_1, V_4 + Z_2, Z_3 \rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_4$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_3 + Z_1 &= [0, \tfrac{\alpha_1 x - yB}{B}, \tfrac{\alpha_1 y + xB}{B}, \tfrac{\alpha_1 z}{B}, \beta_1 u], \\ V_4 + Z_2 &= [\tfrac{zA}{B}, \tfrac{x(\alpha_2 t - zA)}{tB}, \tfrac{y(\alpha_2 t - zA)}{tB}, \tfrac{(x^2+y^2)A + \alpha_2 tz}{tB}, \beta_2 u], \\ Z_3 &= [0, \tfrac{x}{B}, \tfrac{y}{B}, \tfrac{z}{B}, \alpha_3 u] \end{aligned} \tag{4.52}$$

where $A = \sqrt{c^2t^2 - 1}$, $\quad B = \sqrt{x^2 + y^2 + z^2}$.

The transversality condition (1.43) of (4.36) with rank three is always satisfied. Since the Lie algebra generated by $\mathcal{L}_4$ is abelian. Therefore, one can find the invariants of $\mathcal{L}_4$ in any order. The invariants of Z_3 are:

$$m_1 = t, m_2 = \frac{y}{x}, m_3 = \frac{z}{x}, m_4 = u\mathrm{e}^{-\alpha_3 \sqrt{x^2+y^2+z^2}} \tag{4.53}$$

The remaining operators can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$\begin{aligned} V_3 + Z_1 &= [0, {m_2}^2 + 1, m_3 m_2, (\beta_1 - \alpha_3\, \alpha_1)\, m_4], \\ V_4 + Z_2 &= [\tfrac{m_3 \sqrt{c^2 {m_1}^2 - 1}}{\sqrt{1 + {m_2}^2 + {m_3}^2}}, 0, \tfrac{\sqrt{c^2 {m_1}^2 - 1}\sqrt{1 + {m_2}^2 + {m_3}^2}}{m_1}, (\beta_2 - \alpha_3\, \alpha_2)\, m_4] \end{aligned} \tag{4.54}$$

Next, the invariants of $V_3 + Z_1$ are

$$n_1 = m_1, n_2 = \frac{m_3}{\sqrt{{m_2}^2 + 1}}, n_3 = m_4 \mathrm{e}^{(\alpha_3\, \alpha_1 - \beta_1)\arctan(m_2)} \tag{4.55}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$V_4 + Z_2 = [\tfrac{n_2\sqrt{c^2{n_1}^2-1}}{\sqrt{1+{n_2}^2}}, \tfrac{\sqrt{1+{n_2}^2}\sqrt{c^2{n_1}^2-1}}{n_1}, (\beta_2 - \alpha_3\,\alpha_2)\,n_3] \tag{4.56}$$

Finally, the invariants of $V_4 + Z_2$ are

$$\frac{1+{n_2}^2}{{n_1}^2}, n_3\left(cn_2 + \frac{\sqrt{1+{n_2}^2}\sqrt{c^2{n_1}^2-1}}{n_1}\right)^{\frac{\alpha_3\,\alpha_2-\beta_2}{c}} \tag{4.57}$$

Writing the invariants (4.57) in terms of the original variables gives the joint invariants of $\mathcal{L}_4$ as

$$\frac{x^2+y^2+z^2}{t^2\,(x^2+y^2)}, ue^{-\alpha_3 B}e^{A_1\,\mathrm{arctan}\left(\frac{y}{x}\right)}\Big(\frac{cz}{\sqrt{x^2+y^2}} + \frac{AB}{t\sqrt{x^2+y^2}}\Big)^{A_2} \tag{4.58}$$

where $A_1 = \alpha_3\,\alpha_1 - \beta_1$, $A_2 = \frac{\alpha_2\alpha_3-\beta_2}{c}$.

Therefore, the invariant transformations are:

$$w = \frac{x^2+y^2+z^2}{t^2\,(x^2+y^2)},\;\; Z(w) = ue^{-\alpha_3 B}e^{A_1\,\mathrm{arctan}\left(\frac{y}{x}\right)}\Big(\frac{cz}{\sqrt{x^2+y^2}} + \frac{AB}{t\sqrt{x^2+y^2}}\Big)^{A_2} \tag{4.59}$$

Thus, using (4.59), equation (4.3) can be reduced to the ODE:

$$4w^2\,(\,c^2 - \,w)\,Z'' - 4\,w^2(1-A_2)Z' - ((A_1^2 + A_2^2)w + \alpha_3^2)Z = 0 \tag{4.60}$$

which can be transformed using the transformation

$$w = c^2 r, \ Z(w) = r^{\frac{c+\sqrt{c^2+\alpha_3{}^2}}{2c}} R(r) \tag{4.61}$$

to the hypergeometric differential equation

$$r(r-1)R'' + \big((\nu+\mu+1)r - \gamma\big)R' + \nu\mu R = 0 \tag{4.62}$$

with $\nu = \frac{(1+iA_1-A_2)c+\sqrt{c^2+\alpha_3{}^2}}{2c}, \mu = \frac{(1-iA_1-A_2)c+\sqrt{c^2+\alpha_3{}^2}}{2c}, \gamma = \frac{c+\sqrt{c^2+\alpha_3{}^2}}{c}$. The solution of (4.62) is given in terms of the hypergeometric function $F(\nu, \mu; \gamma; r)$ as

$$R(r) = c_1 F(\nu, \mu; \gamma; r) + c_2 r^{1-\gamma} F(\nu-\gamma+1, \mu-\gamma+1; 2-\gamma; r) \tag{4.63}$$

Therefore, the solution of equation (4.60) is

$$Z(w) = \left(\tfrac{w}{c^2}\right)^{\frac{\gamma}{2}} R\left(\tfrac{w}{c^2}\right) \tag{4.64}$$

Thus, the invariant solution of (4.3) is

$$\begin{aligned} u(t,x,y,z) = Z\left(\tfrac{x^2+y^2+z^2}{t^2(x^2+y^2)}\right) e^{\alpha_3\sqrt{x^2+y^2+z^2}} e^{ci(\mu-\nu)\arctan\left(\frac{y}{x}\right)} \\ \left(\tfrac{t\sqrt{x^2+y^2}}{ctz+\sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}}\right)^{\gamma-c\mu+(c-2)\nu} \end{aligned} \tag{4.65}$$

where $\alpha_3^2 = c^2(\gamma-1)^2 - c^2$.

4.4.1.5 Case $\mathcal{L}_5 = \langle V_4 + Z_1, V_5, Z_3 \rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_5$ in Cartesian coordinates are as follows:

$$\begin{aligned} V_4 + Z_1 &= [\tfrac{zA}{B}, \tfrac{x(\alpha_1 t - zA)}{tB}, \tfrac{y(\alpha_1 t - zA)}{tB}, \tfrac{\left(x^2+y^2\right)A + \alpha_1 tz}{tB}, \beta_1 u], \\ V_5 &= [\tfrac{yA}{B}, -\tfrac{yxA}{tB}, \tfrac{\left(x^2+z^2\right)A + cztB}{tB}, -\tfrac{y(zA+ctB)}{tB}, 0], \\ Z_3 &= [0, \tfrac{x}{B}, \tfrac{y}{B}, \tfrac{z}{B}, \alpha_3 u] \end{aligned} \tag{4.66}$$

where $A = \sqrt{c^2t^2 - 1}$, $B = \sqrt{x^2 + y^2 + z^2}$.

The transversality condition (1.43) of (4.66) with rank three is always satisfied. Since the derived Lie algebra generated by $\mathcal{L}_5$ is $\langle V_5 \rangle$. Therefore, one can find the invariants of $\mathcal{L}_5$ by starting with V_5. The invariants of V_5 are:

$$m_1 = xt, m_2 = \sqrt{x^2 + y^2 + z^2}, m_3 = ctz + \sqrt{c^2t^2 - 1}\sqrt{x^2 + y^2 + z^2}, m_4 = u \tag{4.67}$$

The remaining operators can be given in terms of the variables $m_i, i = 1, \ldots, 4$ as

$$\begin{aligned} V_4 + Z_1 &= [\tfrac{\alpha_1 m_1}{m_2}, \alpha_1, \tfrac{m_3(\alpha_1 + cm_2)}{m_2}, \beta_1 m_4], \\ Z_3 &= [\tfrac{m_1}{m_2}, 1, \tfrac{m_3}{m_2}, \alpha_3 m_4] \end{aligned} \tag{4.68}$$

Next, the invariants of $V_4 + Z_1$ are

$$n_1 = \frac{m_2}{m_1}, n_2 = \frac{m_3}{m_1}\mathrm{e}^{-\frac{c}{\alpha_1}m_2}, n_3 = m_4\mathrm{e}^{-\frac{\beta_1}{\alpha_1}m_2} \tag{4.69}$$

In terms of the variables $n_i, i = 1, 2, 3$, the remaining operator is

$$Z_3 = [0, -\tfrac{c}{\alpha_1} n_2, \tfrac{(\alpha_3\,\alpha_1 - \beta_1)}{\alpha_1} n_3] \tag{4.70}$$

Finally, the invariants of Z_3 are

$$n_1, n_3\; n_2^{\frac{\alpha_3\,\alpha_1 - \beta_1}{c}} \tag{4.71}$$

Writing the invariants (4.71) in terms of the original variables gives the joint invariants of $\mathcal{L}_5$ as

$$\frac{\sqrt{x^2 + y^2 + z^2}}{xt}, u\mathrm{e}^{-\alpha_3 B}\left(\frac{czt + A\ B}{xt}\right)^{A_1} \tag{4.72}$$

where $A_1 = \frac{\alpha_3\,\alpha_1 - \beta_1}{c}$.

Therefore, the invariant transformations are:

$$w = \frac{\sqrt{x^2 + y^2 + z^2}}{xt},\;\; Z(w) = u\mathrm{e}^{-\alpha_3 B}\left(\frac{czt + A\ B}{xt}\right)^{A_1} \tag{4.73}$$

Thus, using (4.73), equation (4.3) can be reduced to the ODE:

$$w^2\,(c^2 - w^2)\,Z'' - (c^2 + 2\,(1 - A_1)\,w^2)\,wZ' - ({\alpha_3}^2 + A_1\,(A_1 - 1)\,w^2)\,Z = 0 \tag{4.74}$$

which can be transformed using the transformation

$$w = c\sqrt{r},\ Z(w) = r^{\frac{c-\sqrt{c^2+\alpha_3{}^2}}{2c}} R(r) \tag{4.75}$$

to the hypergeometric differential equation

$$r(r-1)R'' + \big((\nu+\mu+1)r - \gamma\big)R' + \nu\mu R = 0 \tag{4.76}$$

with $\nu = \frac{2\,c-cA_1-\sqrt{c^2+\alpha_3{}^2}}{2c}, \mu = \frac{c-cA_1-\sqrt{c^2+\alpha_3{}^2}}{2c}, \gamma = \frac{c-\sqrt{c^2+\alpha_3{}^2}}{c}$.

The solution of (4.76) is given in terms of the hypergeometric function $F(\nu, \mu; \gamma; r)$ as

$$R(r) = c_1 F(\nu, \mu; \gamma; r) + c_2 r^{1-\gamma} F(\nu - \gamma + 1, \mu - \gamma + 1; 2 - \gamma; r) \tag{4.77}$$

Therefore, the solution of (4.74) is

$$Z\left(w\right) = w^{\gamma}\Big(c_1 F(\nu, \nu - \tfrac{1}{2}; \gamma; \tfrac{w^2}{c^2}) + c_2 w^{2-2\gamma} F(\nu - \gamma + 1, \nu - \gamma + \tfrac{1}{2}; 2 - \gamma; \tfrac{w^2}{c^2})\Big) \tag{4.78}$$

where F is the hypergeometric function. Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\Big(\frac{\sqrt{x^2+y^2+z^2}}{xt}\Big)\ \mathrm{e}^{c\sqrt{\gamma^2-2\,\gamma}\sqrt{x^2+y^2+z^2}}\left(\frac{xt}{czt+\sqrt{x^2+y^2+z^2}\sqrt{c^2t^2-1}}\right)^{\gamma-2\nu+1}. \tag{4.79}$$

4.4.1.6 Case $\mathcal{L}_6 = \langle V_5 + Z_1, V_6 + Z_2, Z_3\rangle$, $Z_3 = V_7 + \alpha_3 V_8$

The generators of $\mathcal{L}_6$ in Cartesian coordinates are as follows:

$$\begin{gathered}
V_5 + Z_1 = [\tfrac{yA}{B}, \tfrac{x(\alpha_1 t - yA)}{tB}, \tfrac{\left(\left(x^2+z^2\right)A+\alpha_1 ty\right)B+ctzB^2}{tB^2}, \tfrac{z(\alpha_1 t - yA)B - cytB^2}{tB^2}, \beta_1 u],\\
V_6 + Z_2 = [\tfrac{xA}{B}, \tfrac{\left(\left(y^2+z^2\right)A+\alpha_2 x t\right)B+ctzB^2}{tB^2}, \tfrac{y(\alpha_2 t - xA)}{tB}, \tfrac{z(\alpha_2 t - xA)B - ctxB^2}{tB^2}, \beta_2 u],\\
Z_3 = [0, \tfrac{x}{B}, \tfrac{y}{B}, \tfrac{z}{B}, \alpha_3 u]
\end{gathered} \tag{4.80}$$

where $A = \sqrt{c^2t^2 - 1}$, $B = \sqrt{x^2 + y^2 + z^2}$.

The transversality condition (1.43) of (4.80) with rank three is always satisfied. One can verify that the joint invariants of $\mathcal{L}_6$ which are:

$$\sqrt{c^2t^2 - 1} + \frac{ctz}{\sqrt{x^2+y^2+z^2}},\ u\mathrm{e}^{-\alpha_3 \sqrt{x^2+y^2+z^2}}\mathrm{e}^{\frac{t(A_1 y + A_2 x)}{\sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}+c\,t\,z}} \tag{4.81}$$

where $A_1 = \alpha_3\,\alpha_1 - \beta_1$, $A_2 = \alpha_3\,\alpha_2 - \beta_2$.

Therefore, the invariant transformations are:

$$w = \sqrt{c^2t^2 - 1} + \frac{ctz}{\sqrt{x^2+y^2+z^2}},\ Z(w) = u\mathrm{e}^{-\alpha_3 \sqrt{x^2+y^2+z^2}}\mathrm{e}^{\frac{t(A_1 y + A_2 x)}{\sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}+c\,t\,z}} \tag{4.82}$$

Thus, using (4.82), equation (4.3) can be reduced to the ODE:

$$c^2w^4\, Z'' + c^2w^3 Z' - \left((c^2 + {\alpha_3}^2)w^2 + {A_1}^2 + {A_2}^2\right)Z = 0 \tag{4.83}$$

which can be transformed using the transform $w = \frac{1}{r}$, $Z(w) = R(r)$ to the modified parametric Bessel equation

$$r^2 R'' + rR' - (\alpha^2 r^2 + v^2)R = 0 \tag{4.84}$$

where $\alpha = \frac{\sqrt{A_1^2+A_2^2}}{c}$, $v = \frac{\sqrt{\alpha_3^2+c^2}}{c}$.

Therefore the solution of (4.83) is:

$$Z\left(w\right) = c_1 \mathrm{K}_v\left(\frac{\alpha}{w}\right) + c_2 \mathrm{I}_v\left(\frac{\alpha}{w}\right)$$

where K_v and I_v are the modified Bessel functions of first and second kind respectively. Thus, the invariant solution of (4.3) is

$$u(t,x,y,z) = Z\left(\sqrt{c^2t^2-1} + \frac{ctz}{\sqrt{x^2+y^2+z^2}}\right) \mathrm{e}^{\alpha_3 \sqrt{x^2+y^2+z^2}} \mathrm{e}^{-\frac{t(A_1 y + A_2 x)}{\sqrt{c^2t^2-1}\sqrt{x^2+y^2+z^2}+c\,t\,z}} \tag{4.85}$$

4.4.2 Non-solvable subalgebras of $\mathcal{L}$

4.4.2.1 Case $\mathcal{L}_1 = \langle V_4, V_5, V_5 - 2cV_2 \rangle$

Since the reduced row echelon form of the operators of $\mathcal{L}_1$ coincides with (2.26), the same argument applies.

4.5 Invariants solutions in spherical coordinates

In this section, we summarize all the obtained invariant solutions of rank one for the wave equation on non-static analog of Einstein spacetime (4.2) as follows:

I) Solution (4.20):

$$u\left(t,r,\theta\right)=\mathrm{e}^{\frac{\beta_1}{\alpha_1}r}Z\left(\left(ct\cos\theta+\sqrt{c^2t^2-1}\right)\mathrm{e}^{-\frac{c\alpha}{\alpha_1}r}\right)\ ,\alpha_1\neq 0, \tag{4.86}$$

where $Z(w)$ is given according to the following cases:

i) $\alpha^2-\alpha_1^2\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{c{\alpha_1}^2+\alpha\,\beta_1+\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{c\left(\alpha^2-{\alpha_1}^2\right)}}+c_2\,w^{\frac{c{\alpha_1}^2+\alpha\,\beta_1-\alpha_1\sqrt{c^2{\alpha_1}^2+2\,c\alpha\,\beta_1+{\beta_1}^2}}{c\left(\alpha^2-{\alpha_1}^2\right)}} \tag{4.87}$$

ii) $\alpha^2-\alpha_1^2=0, c\alpha+\beta_1\neq 0$:

$$Z\left(w\right)=c_1\,w^{\frac{\beta_1^2}{2c\alpha\,\left(c\alpha+\beta_1\right)}} \tag{4.88}$$

II) Solution (4.26):

$$u\left(t,r,\theta\right)=\left(ct\cos\theta+\sqrt{c^2t^2-1}\right)^{\frac{\beta_1}{\alpha\,c}}Z\left(r\right),\ \ \alpha_1=0 \tag{4.89}$$

where

$$Z\left(w\right)=w^{-A_1}\left(c_1\,\mathrm{e}^{c\sqrt{2A_1+A_1^2}w}+c_2\,\mathrm{e}^{-c\sqrt{2A_1+A_1^2}w}\right) \tag{4.90}$$

III) Solution (4.35):

$$u\left(t,r,\theta\right)=\mathrm{e}^{\frac{\beta_1}{\alpha_1}r}Z\left(c\,t\cos\theta+\sqrt{c^2t^2-1}\right),\ \ \alpha_1\neq 0 \tag{4.91}$$

where

$$Z\left(w\right)=c_{1}\,w^{\frac{-c\alpha_{1}+\sqrt{\alpha_{1}^{2}c^{2}+\beta_{1}^{2}}}{\alpha_{1}c}}+c_{2}\,w^{\frac{-c\alpha_{1}-\sqrt{\alpha_{1}^{2}c^{2}+\beta_{1}^{2}}}{\alpha_{1}c}} \tag{4.92}$$

IV) Solution (4.45):

$$u\left(t,r,\theta\right)=\mathrm{e}^{\frac{\beta_{1}}{\alpha_{1}}r}Z\left(\left(\sqrt{c^{2}t^{2}-1}+ct\cos\theta\right)\mathrm{e}^{-\frac{c}{\alpha_{1}}r}\right),\ \alpha_{1}\neq 0, \tag{4.93}$$

where $Z(w)$ is given according to the following cases:

i) $\alpha_1^2-1\neq 0$:

$$Z\left(w\right)=c_{1}\,w^{\frac{-c{\alpha_{1}}^{2}-\beta_{1}+\alpha_{1}\sqrt{c^{2}{\alpha_{1}}^{2}+2\,c\,\beta_{1}+{\beta_{1}}^{2}}}{c\left({\alpha_{1}}^{2}-1\right)}}+c_{2}\,w^{\frac{-c{\alpha_{1}}^{2}-\beta_{1}-\alpha_{1}\sqrt{c^{2}{\alpha_{1}}^{2}+2\,c\,\beta_{1}+{\beta_{1}}^{2}}}{c\left({\alpha_{1}}^{2}-1\right)}} \tag{4.94}$$

ii) $\alpha_1^2-1=0, c+\beta_1\neq 0$:

$$Z\left(w\right)=c_{1}\,w^{\frac{\beta_{1}^{2}}{2c\,(c+\beta_{1})}} \tag{4.95}$$

V) Solution (4.51):

$$u\left(t,r,\theta\right)=\left(ct\cos\theta+\sqrt{c^{2}t^{2}-1}\right)^{\frac{\beta_{1}}{c}}\,Z\left(r\right) \tag{4.96}$$

where

$$Z\left(w\right)=w^{-\frac{\beta_{1}}{c}}\left(c_{1}\,\mathrm{e}^{\sqrt{{\beta_{1}}^{2}+2\,\beta_{1}c}\;w}+c_{2}\,\mathrm{e}^{-\sqrt{{\beta_{1}}^{2}+2\,\beta_{1}c}\;w}\right) \tag{4.97}$$

VI) Solution (4.65):

$$u\,(t,r,\theta)=\mathrm{e}^{\alpha_3\,r}\mathrm{e}^{ci(\mu-\nu)\theta}\left(\frac{t\sin\theta}{ct\cos\theta+\sqrt{c^2t^2-1}}\right)^{\gamma-c\mu+(c-2)\nu}Z\left(\frac{\csc^2\theta}{t^2}\right) \tag{4.98}$$

where

$$Z(w)=\left(\tfrac{w}{c^2}\right)^{\frac{\gamma}{2}}\Big(c_1F(\nu,\mu;\gamma;\tfrac{w}{c^2})+c_2(\tfrac{w}{c^2})^{1-\gamma}F(\nu-\gamma+1,\mu-\gamma+1;2-\gamma;\tfrac{w}{c^2})\Big), \tag{4.99}$$

F is the hypergeometric function and $\alpha_3^2=c^2(\gamma-1)^2-c^2$.

VII) Solution (4.79):

$$u\,(t,r,\theta)=\mathrm{e}^{\alpha_3 r}\left(\frac{t\cos\varphi\sin\theta}{ct\cos\theta+\sqrt{c^2t^2-1}}\right)^{\gamma-2\nu+1}Z\left(\frac{1}{t\cos\varphi\sin\theta}\right) \tag{4.100}$$

where

$$Z\,(w)=w^{\gamma}\Big(c_1F(\nu,\nu-\tfrac{1}{2};\gamma;\tfrac{w^2}{c^2})+c_2w^{2-2\gamma}F(\nu-\gamma+1,\nu-\gamma+\tfrac{1}{2};2-\gamma;\tfrac{w^2}{c^2})\Big) \tag{4.101}$$

and F is the hypergeometric function.

VIII) Solution (4.85):

$$\begin{aligned}u\,(t,r,\theta,\varphi)=&\frac{1}{\sqrt{c^2t^2-1}+c\,t\,\cos\theta}\mathrm{e}^{\alpha_3 r}\mathrm{e}^{-\left(\frac{A_1\,\sin\varphi\sin\theta+A_2\,\cos\varphi\sin\theta}{\sqrt{c^2t^2-1}+c\,t\,\cos\theta}\right)t}\\&Z\left(\sqrt{c^2t^2-1}+c\,t\,\cos\theta\right)\end{aligned} \tag{4.102}$$

where

$$Z\left(w\right)=c_{1}\mathrm{K}_{v}\left(\alpha w\right)+c_{2}\mathrm{I}_{v}\left(\alpha w\right), \tag{4.103}$$

and K_v and I_v are the modified Bessel functions of first and second kind respectively.

IX) The solution that comes from the non-solvable algebra is

$$u\left(t,r,\theta,\varphi\right)=c_{1}+c_{2}\ t \tag{4.104}$$

REFERENCES

[1] Ali, S., Azad, H., Biswas, I., Ghanam, R. and Mustafa, M. T. Embedding algorithms and applications to differential equations, J. Symbolic Computation (to appear).

[2] Azad, H., Al-Dweik, A. Y., Mahomed, F. M. and Mustafa, M. T. A point symmetry based method for transforming ODEs with three-dimensional symmetry algebras to their canonical forms, Applied Mathematics and Computation 289 (2016), 444-463.

[3] Azad, H., Al-Dweik, A. Y., Ghanam, R. and Mustafa, M. T. Symmetry analysis of wave equation on static spherically symmetric spacetimes with higher symmetries, J. Math. Phys. 54 (2013), 063509, 23.

[4] Azad, H. and Biswas, I. A note on real algebraic groups, Forum Mathematicum (2016), 539-543.

[5] Azad, H., Biswas, I., Ghanam, R. and Mustafa, M. T. On computing joint invariants of vector fields, J. Geometry and Physics 97 (2015), 69-76.

[6] Azad, H., and Ziad, M. Spherically symmetric manifolds which admit five isometries, J. Math. Phys. 36 (1995), 1908-1911.

[7] Azad, H., Biswas, I. and Chatterjee, P. On the maximal solvable subgroups of semisimple algebraic groups, J. Lie Theory 22 (2012), 1169–1179.

[8] Bluman, G. and Stephen, C. A. Symmetry and Integration Methods for Differential Equations, Springer Science and Business Media (2002).

[9] Bokhari, A. H. and Qadir, A. Symmetries of static, spherically symmetric space-times, J. Math. Phys. 28 (1987), 1019–1022.

[10] Bozhkov, Y. and Freire, I. L. Special conformal groups of a Riemannian manifold and Lie point symmetries of the nonlinear Poisson equation, J. Differential Equations 249 (2010), 872-913.

[11] De Graaf, W. A. Lie Algebras: Theory and Algorithms, North-Holland Mathematical Library (2000).

[12] Hilgert, J. and Neeb, K. H. Structure and Geometry of Lie Groups, Springer Science and Business Media (2011).

[13] Hydon, P. E. Symmetry methods for differential equations: A beginner's guide, Cambridge University Press (2000).

[14] Ibragimov, N. H. Optimal systems of subgroups and classification of invariant solutions of equations for planar non-stationary gas flows, Master of Science

Thesis in Mathematics, Institute of Hydrodynamics, USSR Acad. Sci. (1965). English translation: Paper 1 in Ibragimov, N.H. Selected Works (2006).

[15] Ibragimov, N. H. Transformation groups applied to mathematical physics, Springer Science and Business Media (2001).

[16] Ibragimov, N. H. Selected works, ALGA publ., BTH, Karlskrona (2006).

[17] Knapp, A. W. Lie groups beyond an introduction, Springer Science and Business Media (2002).

[18] Olver, P. J. Applications of Lie groups to differential equations, Springer Science and Business Media (2000).

[19] Ovsiannikov, L. V. Group analysis of differential equations, Acad. Press (1982).

[20] Ovsiannikov, L. V. Group properties of differential equations, Siberian Branch, USSR Academy of Sciences (1962).

[21] Qadir, A. and Ziad, M. The classification of spherically symmetric space-times, Il Nuovo Cimento B 110 (1995), 317-334.

[22] Snobl, L. and Winternitz, P. Classification and identification of Lie algebras, American Mathematical Soc. 33 (2014).

[23] Winitzki, S. Linear algebra via exterior products, https://sites.google.com/site/winitzki/linalg (2010).

Vitae

- Name: Khaleel Ibrahim Khaleel Anaya
- Nationality: Palestinian
- Date of Birth: Dec 1, 1992
- Email: khalil.anaya1@gmail.com
- Present Adress: Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, P.O. Box 7722, Dhahran 31261, Saudi Arabia.
- Permanent Address: Azzoun, Palestine.

Qualifications:

- Master Degree in Mathematics Statistics from King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia in May 2017.
- Bachelor Degree in Mathematics from An-Najah National University, Nablus, Palestin in July 2013.

Practical Experiences:

- September 2013- June 2014: Math teacher at Abdullah Ibn Abbas schools in Palestine.

- September 2014- January 2015: Math teacher at the ministry of education in Palestine.

References:

- Dr. Ahmed Yousuf Al-Dweik (aydweik@kfupm.edu.sa)

  Associate Professor

  Department of Mathematics and Statistics

  KFUPM

- Dr. Messaoudi, S. A. (messaoud@kfupm.edu.sa)

  Distinguished Professor

  Department of Mathematics and Statistics

  KFUPM

- Dr. Jawad Younes Abuihlail (abuhlail@kfupm.edu.sa)

  Professor

  Department of Mathematics and Statistics

  KFUPM